The Spectral Index of Brauer Classes

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Chapter 1

Introduction

In this thesis, I use the étale K-theory of twisted sheaves to study the representability of elements α of the Brauer group of a scheme U by Azumaya algebras. The main tools are K-theory, stacks of symmetric monoidal categories, and the stable homotopy groups of classifying spaces of finite groups. The results include a lower bound, $spi(\alpha)$, on the index of an Azumaya algebra representing α. The integer $spi(\alpha)$, the spectral index, is defined cohomologically as a permanent cycle in a certain spectral sequence. The spectral sequence is also the smallest non-zero rank of an element of $\mathbf{K}_0^{\alpha,\text{ét}}$. That is, in twisted étale K-theory, there is an α-twisted vector bundle of rank $spi(\alpha)$.

The α-twisted sheaves play an important role in the study of α because they furnish a nice category (even stack) of objects each of whose associated endomorphism sheaf is an Azumaya algebra in the class α. Thus, finding smallest rank α-twisted sheaves is the same as finding smallest rank Azumaya algebras representing α. The advantage of using the category of twisted sheaves is that it provides a good category for the application of K-theory. In fact, the stack of twisted sheaves used here is a stack of symmetric monoidal and exact categories. The descent spectral sequence of the associated K-theory presheaf furnishes the results.

1.1 Background

Let U be a scheme. The general area to which this thesis contributes is the problem of determining, for $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$, the smallest rank of an Azumaya algebra representing α. Any solution to this problem leads to potential geometric insight into the study of U and α. For the remainder of this section, I concentrate on the case where U is the spectrum of a field. Everything may be expressed using division algebras in this case, and the problem becomes find the rank of the unique division algebra in the class α.

If k is a field and A is a central simple algebra over k, then A is a projective k-module of square rank n^2. Define the index of A to be $ind(A) = n$. If D is a division algebra central over k, then the period of D, $per(D)$, is the order of the class $[D]$ of D in the Brauer group $\mathrm{Br}(k)$. Similarly, if $\alpha \in \mathrm{Br}(k)$, write $per(\alpha)$ for its order, and define $ind(\alpha)$ to be the index of the unique division algebra having class α.

For $\alpha \in \mathrm{Br}(k)$,

$$per(\alpha)|ind(\alpha),$$

and moreover the two integers have the same set of prime divisors. However, in general, there are no more constraints on the pair of integers. Indeed, for any prime l, and any positive integers $e < f$, there is a field k and a division algebra D central over k with

$$per(D) = l^e$$
$$ind(D) = l^f.$$

On the other hand, for a fixed field k, the situation is often considerably more restricted. Indeed, the period and index coincide for all division algebras over fields of the following types:

- p-adic fields, by class field theory;
- number fields, by the Brauer-Hasse-Noether theorem;
- C_2-fields, when $per(\alpha) = 2^a 3^b$, by Artin and Harris [1];
- function fields $k(X)$ of algebraic surfaces X over an algebraically closed field k, by de Jong [11];
- quotient fields K of excellent henselian two-dimensional local domains with residue field k separably closed when α is a class of period prime to the characteristic of k, by Colliot-Thélène , Ojanguren, and Parimala [9];
- fields $l((t))$ of transcendence degree 1 over l, a characteristic zero field of cohomological dimension 1, by Colliot-Thélène , P. Gille, and Parimala [8].

Note that in some sense these fields are all 2-dimensional.

Saltman [25] showed that

$$ind(D)|per(D)^2$$

holds for division algebras over the function fields of curves over p-adic fields. Lieblich, in [22] has shown that this is also true for the function fields of surfaces over finite fields. Finally, Lieblich and Krashen have established in [21] the sharp relation

$$ind(D)|per(D)^d$$

for the function fields of curves over d-local fields, such as $k((t_1))\cdots((t_d))$, where k is algebraically closed. Moreover, in these examples, the exponent is the best possible.

These examples should suffice to establish the reasonableness of a conjecture, due to Colliot-Thélène .

Conjecture 1.1.1 (Period-Index Conjecture). *If k is a field of dimension d, then*

$$ind(\alpha)|(per(\alpha))^{d-1}$$

for all $\alpha \in \mathrm{Br}(k)$.

In the conjecture, the dimension should mean either that k is C_d, that k is the function field of a d-dimensional algebraic variety over an algebraically closed field, that k is the function field of a $(d-1)$-dimensional variety over a finite field, or that k is the function field of a $(d-2)$-dimensional variety over a local field.

In general, the conjecture is known to be false if dimension is taken to be the cohomological dimension of the field. For prime powers l^e and l^f, with $e \leq f$, a construction of Merkurjev [23] can be used to construct a field k with $cd_l(k) = 2$, and a division algebra D over k with $per(D) = l^e$ and $ind(D) = l^f$.

1.2 Summary of Results

Theorem 1.2.1. *Let U be a geometrically connected quasi-separated scheme of finite étale cohomological dimension d, and let $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)_{\text{tors}}$. Then, there is a unique integer $spi(\alpha)$ having the following properties:*

1. *in the spectral sequence*
$$\mathrm{H}^s(U_{\text{ét}}, \mathcal{K}_t^\alpha) \Rightarrow \mathbf{K}_{t-s}^{\alpha,\text{ét}}(U),$$
the integer $spi(\alpha) \in \mathbb{Z} \cong \mathrm{H}^0(U_{\text{ét}}, \mathcal{K}_0^\alpha)$ is the smallest such that $d_k^\alpha(spi(\alpha)) = 0$ for all $k \geq 2$, where d_k^α is the kth differential in the spectral sequence;

2. ***divisibility**: $per(\alpha)|spi(\alpha)$;*

3. ***obstruction**: if $\mathcal{A}$ is an Azumaya algebra in the class of α, then $spi(\alpha)|ind(\mathcal{A})$;*

4. ***bound**:*
$$spi(\alpha)| \prod_{j\in\{1,\ldots,d-1\}} l_j^\alpha.$$
where l_j^α is the least common multiple of the exponents of π_j^s and $\pi_j^s(B\mathbb{Z}/(per(\alpha)))$.

In particular, $spi(\alpha)$ is finite even if α is not representable by an Azumaya algebra.

The first property serves as a definition and gives uniqueness. The **divisibility** property is proven in Theorem 4.13.3, the **obstruction** property in Theorem 3.2.5, and the **bound** property in Theorem 3.3.4. An analysis of the integers l_j^α, together with the **divisibility** and **bound** properties above and the fact that the period and index have the same prime divisors for Brauer classes on a field, gives the following, Theorem 3.3.8:

Theorem 1.2.2 (Period-Spectral Index Theorem)**.** *Let k be a field, and let $\alpha \in \mathrm{H}^2(k, \mathbb{G}_m)$. Let S be the set of prime divisors of $per(\alpha)$, and suppose that $d = cd_S k$ satisfies $d = 2c$ or $d = 2c+1$ and that $d < 2\min_{q \in S}(q)$. Then,*

$$spi(\alpha) | (per(\alpha))^c.$$

The theorem should be viewed as a cohomological version of the period-index conjecture.

Remark 1.2.3. The reader is warned of two things. First, I do not know if $spi(\alpha)$ is ever distinct from $per(\alpha)$. Second, I do not know if the restriction on the primes in Theorem 1.2.2 is necessary, although it is necessary for my proof. The first problem would be solved if one proved the following conjecture.

Conjecture 1.2.4. *Let $k = \mathbb{C}((t_1)) \cdots ((t_d))$ be an iterated Laurent series field over the complex numbers. Then, for $\alpha \in \mathrm{Br}(k)$,*

$$spi(\alpha) = ind(\alpha).$$

In particular, if $d \geq 4$, then, for any prime l, and $D = (t_1, t_2)_{\zeta_l} \otimes_k (t_3, t_4)_{\zeta_l}$,

$$l = per(D) < spi(D) = ind(D) = l^2.$$

A similar statement holds for d-local fields.

One reason to believe this conjecture is that for d-local fields, Becher and Hoffman have established [4] that the index satisfies

$$ind(\alpha) | per(\alpha)^c,$$

for d-local fields k, where $d = 2c$ or $d = 2c + 1$. This bound is the same as given for the spectral index.

1.3 Methods

In Chapter 2, the necessary technicalities for defining the spectral index and proving the **obstruction** and **bound** properties are established. The chapter gives details on stacks of symmetric monoidal categories and how to twist these stacks by 2-cocycles of natural transformations. In particular, if $\mathbf{Proj}$ is the stack of locally free and finite rank coherent $\mathcal{O}_U$-modules, and if $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$, then the methods outlined in the chapter give a new stack $\mathbf{Proj}^\alpha$, the stack of α-twisted coherent projective modules. A key point which is emphasized there is that the twisting is functorial. Suppose that α lifts to $\beta \in \mathrm{H}^2(U_{\text{ét}}, \mu_n)$. Let $\mathbf{nSets}$ be the stack of sheaves of finite and faithful μ_n-sets. This is the free stack of symmetric monoidal categories generated by the trivial μ_n-gerbe of μ_n-torsors. The class β twists $\mathbf{nSets}$ to create $\mathbf{nSets}^\beta$. Because twisting is functorial, there is a functor of stacks of symmetric monoidal categories

$$\mathbf{nSets}^\beta \to \mathbf{Proj}^\alpha,$$

called the twisted unit morphism. Let $\mathbf{T}^\beta \to \mathbf{K}^\alpha$ be the associated maps of presheaves of spectra, given by applying the K-theory functor to the symmetric monoidal categories on each object in the étale site on U. The morphism is called the twisted unit morphism because if α is trivial, so that $n = 1$, the map becomes the familiar unit morphism

$$S \to \mathbf{K},$$

where S is the constant presheaf on the sphere spectrum, and $\mathbf{K}$ is the usual $\mathbf{K}$-theory presheaf.

The **obstruction** property is then proven by examining the descent spectral sequence associated to $\mathbf{K}^\alpha$, while the **bound** property is reduced to analyzing the stalks of the homotopy sheaves of $\mathbf{T}^\beta$. These stalks are stably equivalent to $S^0 \vee B\mathbb{Z}/(n)$. This is carried out in Chapter 3.

Chapter 4 includes the details necessary to establish the **divisibility** property. The main tools are a Čech version of the descent spectral sequence and an analysis of the differentials in that spectral sequence. After completing the work in this chapter, I became aware of an equivalent proposition [20, Proposition 6.9.1] in a preprint of Kahn and Levine. Essentially, one must show that in the descent spectral sequence for $\mathbf{K}^\alpha$,

$$d_2^\alpha(1) = \alpha$$

via the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(U_{\text{ét}}, \mathcal{K}_0^\alpha) & \longrightarrow & \mathbb{Z} \\ {\scriptstyle d_2^\alpha}\downarrow & & \downarrow \\ \mathrm{H}^2(U_{\text{ét}}, \mathcal{K}_1^\alpha) & \longrightarrow & \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m), \end{array}$$

where the horizontal arrows are natural isomorphisms. By the definition of the spectral index, it follows that it must be divisible by the period of α.

1.4 Future Directions

Evidently, the most important thing to do next is to establish that the spectral index and the period are distinct invariants of Brauer classes. I am currently working to establish Conjecture 1.2.4, which would suffice to do this.

In another direction, I would like to explore the topological consequences of the period-spectral index theorem. Suppose that U is an algebraic variety over $\mathbb{C}$, and suppose for simplicity that

$$\mathrm{H}^2(U_{\mathrm{an}}, \mathcal{O}_U) = (0).$$

Then, there is an inclusion

$$0 \to \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m) \xrightarrow{\simeq} \mathrm{H}^2(U_{\mathrm{an}}, \mathcal{O}_U^*) \xrightarrow{t} \mathrm{H}^3(U_{\mathrm{an}}, \mathbb{Z})_{\mathrm{tors}}.$$

Given Thomason's theorem that

$$\mathbf{K}^{\text{ét}}(U, \mathbb{Z}/(l^v)) \xrightarrow{\simeq} \mathbf{K}^{\mathrm{top}}(U, \mathbb{Z}/(l^v)),$$

it is natural to guess that

$$\mathbf{K}^{\alpha,\text{ét}}(U) \to \mathbf{K}^{\alpha}(U)_{\mathbf{K}^{\mathrm{top}}} \xrightarrow{\simeq} \mathbf{K}^{t(\alpha),\mathrm{top}}(U)$$

is a covering followed by an equivalence, where the middle term is the $\mathbf{K}^{\mathrm{top}}$-localization of the presheaf $\mathbf{K}^{\alpha}$. If this is true, then there is a $t(\alpha)$-twisted topological bundle $\mathcal{E}$ of rank $spi(\alpha)$.

The **bound** property of the spectral index then would provide an *a priori* upper bound on the smallest ranks of $t(\alpha)$-twisted topological bundles. This sort of result seems like it would be entirely new.

Of particular interest is the fact that even if $spi(\alpha) = per(\alpha)$ in all cases, this would say something interesting about the existence of $t(\alpha)$-twisted vector bundles: that a $t(\alpha)$-twisted vector bundle of rank $spi(\alpha) = per(\alpha)$ exists.

Once it has been established that the spectral index is a new invariant, I will study bicyclic algebras. These are tensor products $D = D_1 \otimes_k D_2$, where

$$D_1 = (a, b)_{\zeta_l}$$
$$D_2 = (c, d)_{\zeta_l},$$

and where ζ_l is a primitive lth root of unity in k. The algebra $(a, b)_{\zeta_l}$ is defined to be $k\{x, y\}$, where $x^l = a$, $y^l = b$, and $xy = \zeta_l yx$. If D has non-trivial Brauer class, then $per(D) = l$, and it is an interesting problem to determine whether or not D is a division algebra.

When $l = 2$, in which case D_1 and D_2 are quaternion algebras, then a criterion of Albert asserts that D is a division algebra if and only if a certain quadratic form defined in terms of a, b, c, d has no non-trivial zeros in k.

For primes besides 2, no analogue of Albert's criterion exists. However, if $spi(D)$ is more than $per(D)$, it would follow that D is in fact a division algebra. Finding a way to compute the spectral index, which is either l or l^2 in this case, would then be of great interest for determining when these bicycle algebras are division.

Chapter 2

The Brauer Group and Sheaves

In this chapter, I recall some technicalities on stacks of symmetric monoidal categories. The basic model of such objects is the stack $\mathbf{Proj}$ of projective modules of finite rank on a scheme U in the étale topology. An element of the Brauer group $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$ twists this stack to create a new stack of symmetric monoidal categories $\mathbf{Proj}^\alpha$ which has an associated K-theory presheaf of spectra $\mathbf{K}^\alpha$. In order to get some handle on the descent properties of twisted etale K-theory, that is the presheaf $\mathbf{K}^{\alpha,\text{ét}}$, it is extremely tempting to try and use a unit morphism like the one in the untwisted case:

$$S \to \mathbf{K}^{\text{ét}},$$

where S is the sphere spectrum. However, if α is non-trivial, no such unit morphism exists. The problem is that the twisting process is not compatible with the unit morphism. However, there is a good alternative candidate in the case that α is a torsion class. Namely, suppose β is a lift of α to $\mathrm{H}^2(U_{\text{ét}}, \mu_n)$, where $per(\alpha) = n$. There is a functor of stacks of symmetric monoidal categories

$$\mathbf{nSets} \to \mathbf{Proj},$$

where $\mathbf{nSets}$ is the stack of sheaves of finite and faithful μ_n-sets. Any such sheaf is a disjoint unions of μ_n-torsors, and the functor sends a μ_n-torsor to the associated locally free and finite rank $\mathcal{O}_U$-module and disjoint unions to direct sums. This functor *is* compatible with the twisting process, and is in some sense initial among such functors to $\mathbf{Proj}$. One therefore obtains a twisted unit morphism

$$\mathbf{nSets}^\beta \to \mathbf{Proj}^\alpha,$$

where $\mathbf{nSets}^\beta$ is the stack of β-twisted sheaves of finite and faithful μ_n-sets.

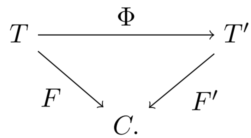
Once the twisted unit morphism has been established, it will be used in the next chapter to examine some differentials in the descent spectral sequence for twisted K-theory in the étale topology. The point is that in a fairly large range, the stalks of the K-theory sheaves $\mathbf{T}_k^\beta$ of $\mathbf{nSets}^\beta$ are easy to compute.

In the final section, a completely intrinsic notion of a stack of symmetric monoidal categories is developed. This is logically independent of the rest of the thesis. The purpose of including the material here is to set both stacks and symmetric monoidal categories in their proper setting of fibered (or cofibered) categories without specifying specific pull-back (or push-forward) functors.

2.1 Fibered Categories

The material in this section is contained in, or easily derived from, [17, Exposé VI].

Let C be any category, and let $F : T \to C$ be a functor. In this case, call T a C-category. A morphism or functor of C-categories is a functor $\Phi : T \to T'$ that commutes with the structure morphisms:



Call such a functor a C-**functor**. There is also the notion of a C-**homomorphism** between two C-functors $\Phi, \Psi : T \to T'$. This is a natural transformation $\omega : \Phi \Rightarrow \Psi$ such that for every object A of T,

$$F'(\Phi(A) \xrightarrow{\omega(A)} \Psi(A)) = Id_{F(A)},$$

as morphisms in C.

In general, if $\phi : U \to V$ is a morphism in C, and if A is an object of T_U and B is an object of T_V, let $Hom_\phi(A, B)$ denote the morphisms f of $Hom_T(A, B)$ such that $F(f) = \phi$.

Definition 2.1.1. A morphism $f : A \to B$ in T is called **cartesian** if, for every morphism $g : A' \to B$ such that $F(g) = F(f)$, there exists a unique $h : A' \to A$ such that $g = f \circ h$.

$$\begin{array}{ccc} A' & & \\ \vdots^{h} & \searrow^{g} & \\ \downarrow & & \\ A & \xrightarrow{f} & B \\ \top & & \top \\ F & & F \\ \downarrow & & \downarrow \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array}$$

In this case, call A a pull-back of B under $F(f) : F(A) \to F(B)$.

For an object V of C, denote by T_V the category consisting of those objects A of T such that $F(A) = V$. The morphisms of T_V are the morphisms a of T such that $F(a) = Id_V$.

Definition 2.1.2. The C-category $F : T \to C$ is called **pre-fibered** if, for every morphism $\phi : V \to W$ in C and every object B in T_W, there is a cartesian morphism $f : A \to B$ such that $F(f) = \phi$. The C-category $F : T \to C$ is called **fibered** if it is pre-fibered and if the composition of cartesian morphisms is cartesian.

Definition 2.1.3. A functor of pre-fibered C-categories is called **cartesian** if it preserves cartesian morphisms.

Definition 2.1.4. Let $F : T \to C$ be a pre-fibered category. A choice of a cartesian pull-back morphism $f^B_\phi : A^B_\phi \to B$ for every $\phi : V \to W$ in C and B in T_W is called a **clivage** for $F : T \to C$. By definition, every fibered category has a clivage. If $f^B_{Id_V} = Id_B$ for all identity morphisms $Id_V : V = V$, then the clivage is said to be **normalized**. The cartesian morphisms that make up the clivage are called the transport morphisms.

Lemma 2.1.5. *Let $F : T \to C$ be a pre-fibered C-category with clivage $f^B_\phi : A^B_\phi \to B$. For $\phi : V \to W$ in C, the clivage uniquely defines a functor $\phi^* : T_W \to T_V$, given on objects by taking the domain of the pull-back maps: $B \mapsto A^B_\phi$. Moreover, for each chain of morphisms $U \xrightarrow{\pi} V \xrightarrow{\phi} W$, there is a natural transformation of functors $\lambda_{\pi,\phi} : \pi^* \circ \phi^* \Rightarrow (\phi \circ \pi)^*$ such that the following diagram of natural transformations commutes for every $T \xrightarrow{\theta} U \xrightarrow{\pi} V \xrightarrow{\phi} W$:*

$$\begin{array}{ccc} \theta^* \circ \pi^* \circ \phi^* & \xrightarrow{\theta^* \circ \lambda_{\pi,\phi}} & \theta^* \circ (\phi \circ \pi)^* \\ {\scriptstyle \lambda_{\theta,\pi} \circ \phi^*}\downarrow & & \downarrow{\scriptstyle \lambda_{\theta,\phi\circ\pi}} \\ (\pi \circ \theta)^* \circ \phi^* & \xrightarrow[\lambda_{\pi\circ\theta,\phi}]{} & (\phi \circ \pi \circ \theta)^* \end{array}$$

If T is fibered, then the $\lambda_{\pi,\phi}$ are natural equivalences.

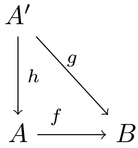
Definition 2.1.6. A normalized clivage f_ϕ^B is called a **scindage** if for all composable ϕ and π in C,

$$\pi^* \circ \phi^* = (\phi \circ \pi)^*,$$

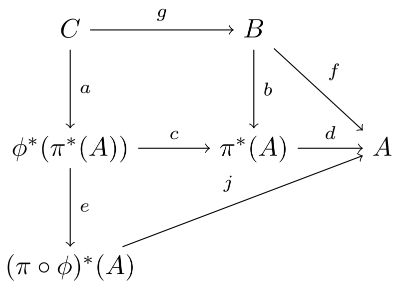
and $\lambda_{\pi,\phi}$ is the identity natural transformation.

Lemma 2.1.7. *Suppose that $F : T \to C$ is a pre-fibered category with scindage f_ϕ^B. Then, T is fibered.*

Proof. Suppose that



where f is cartesian and h is an isomorphism. Then, h is cartesian. Let $\pi : V \to W$ and $\phi : U \to V$, and let A be an object of T_W. Suppose that $f : B \to A$ is a cartesian π-morphism and $g : C \to B$ is a cartesian ϕ-morphism. Then, there is a unique commutative diagram



where a and b are unique isomorphisms guaranteed to exist by the cartesian hypothesis, where c, d, and j are the transport morphisms, and where e is the identity guaranteed by the scindage condition. It follows that $f \circ g$ is isomorphic to the cartesian morphism f. This shows that compositions of cartesian morphisms are cartesian. □

Let (Fib/C) denote the category of fibered C-categories and cartesian functors. In fact, (Fib/C) naturally has the structure of a strict 2-category in which the 2-morphisms are the C-homomorphisms of C-functors. Define (Fib_{cliv}/C) to be the category of fibered C-categories with a fixed normalized clivage. The morphisms are the cartesian functors which take transport morphisms to transport morphisms. Similarly, define (Fib_{scind}/C) to be the full sub-category of (Fib_{cliv}/C) given by the fibered C-categories with clivage where the clivage is actually a scindage.

Let $\Phi : T \to T'$ be a morphism in (Fib/C), and let $f_\phi^B : A_\phi^B \to B$ be a clivage for T and $g_\phi^D : C_\phi^D \to D$ be a clivage for T'. In general, for $\phi : V \to W$ in C, the diagram

$$\begin{array}{ccc} T_V & \xrightarrow{\Phi} & T'_V \\ \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} \\ T_W & \xrightarrow{\Phi} & T'_W \end{array}$$

is not commutative. But, there are unique inverse natural transformations $\Phi \circ \phi^* \Rightarrow \phi^* \circ \Phi$ and $\phi^* \circ \Phi \Rightarrow \Phi \circ \phi^*$.

Given a clivage f_ϕ^B for T, the collection of morphisms $\Phi(f_\phi^B)$ may be expanded into a clivage for T'. Indeed, all of the morphisms are cartesian by hypothesis. For each D in the image of Φ, choose exactly one $\Phi(f_\phi^B)$ for each ϕ. Then, for all D not in the image, choose an arbitrary cartesian morphism for each ϕ. This defines a clivage. For this clivage, the diagram above is commutative. Using a similar argument, for any finite diagram of fibered categories in (Fib/C), they may be simultaneously equipped with clivage so that all morphisms commute with pull-back.

Let $Aut(1_{T/C})$ denote the group of C-automorphisms of the identity C-functor $1 : T \to T$. This can be pre-sheafified on C. For an object V of C, let $C_{/V}$ denote the arrow category of objects over V in C. If $F : T \to C$ is a fibered C-category, then we can restrict this to a fibered $C_{/V}$-category $F_{/V} : T_{/V} \to C_{/V}$ simply by letting $F_{/V}$ be the full subcategory of objects and morphisms sent into $C_{/V}$ by F. Then, $F_{/V} : T_{/V} \to C_{/V}$ is a fibered category over $C_{/V}$. This defines a functor

$$(Fib/C) \to (Fib/C_{/V}).$$

Now, define a presheaf

$$\underline{\mathrm{Aut}}(1_{T/C}) : C^{\mathrm{op}} \to (Groups)$$

by

$$V \mapsto \mathrm{Aut}(1_{T_{C_{/V}}/C_{/V}}).$$

Call this is the presheaf of fibered automorphisms of the identity of T.

Definition 2.1.8. Let $G : C^{\mathrm{op}} \to (Groups)$ be a presheaf of groups on C. An object $F : T \to C$ of (Fib/C) together with a fixed morphism of presheaves of groups $G \to \underline{\mathrm{Aut}}(1_{T/C})$ is called a **G-fibered C-category**. A morphism of G-fibered C-categories is a cartesian morphism of C-categories that preserves the action of G. To be specific, if $\Phi : T \to T'$ is a cartesian morphism of fibered C-categories, then Φ is a morphism of G-fibered C-categories if for every V in C, every $g \in G(V)$, and every object A of $T_{/V}$,

$$\Phi(g(A)) = g(\Phi(A)),$$

as automorphisms of $\Phi(A)$.

Let $(G{-}Fib/C)$ be the category of G-fibered C-categories and morphisms of G-fibered C-categories.

2.2 Categories Fibered in Symmetric Monoidal Categories

For the definitions of symmetric monoidal category and symmetric monoidal functor, see [27, Section 1], where symmetric monoidal categories go under the name tensor categories.

Definition 2.2.1. An object $F : T \to C$ of (Fib/C) is said to **fibered in symmetric monoidal categories** if it is equipped with the structure of a symmetric monoidal category on each fiber category T_V for V in C and if every choice of clivage, the pull-backs

$$\phi^* : T_V \to T_W$$

are in fact a strong monoidal functor. Thus, it is a symmetric monoidal functor such that the coherence maps

$$\phi_{B_0,B_1} : \phi^*(B_0) \otimes \phi^*(B_1) \to \phi^*(B_0 \otimes B_1),$$

and

$$\phi_1 : 1_{T_W} \to \phi^*(1_{T_V})$$

are isomorphisms.

Let $(Fib^{\otimes}/C)$ denote the category of fibered C-categories that are fibered in symmetric monoidal categories together with cartesian functors that induce strong symmetric monoidal functors on the fiber categories. Just as above, for an object $F : T \to C$ of $(Fib^{\otimes}/C)$ there is a presheaf of fibered monoidal automorphisms

$$\underline{\mathrm{Aut}}^{\otimes}(1_{T/C}).$$

A category $F : T \to C$ fibered in symmetric monoidal categories together with a presheaf morphism $G \to \underline{\mathrm{Aut}}^{\otimes}(1_{T/C})$ is called a G-fibered C-category fibered in symmetric monoidal categories. Denote the category of these by

$$(G{-}Fib^{\otimes}/C).$$

Example 2.2.2. Let $\mathbf{Mod}(X) \to X_{Zar}$ denote the fibered category of quasi-coherent $\mathcal{O}_X$-modules on the Zariski site of X. Then, $\mathbf{Mod}(X)$ is naturally a $\mathbb{G}_{m,X}$-fibered X_{Zar}-category. But, $\mathbf{Mod}(X)$ naturally has a symmetric monoidal category structure using direct sum. One may check that $\mathbf{Mod}(X)$ defines an object of $(\mathbb{G}_{m,X}\text{–}Fib^{\otimes}/X_{Zar})$.

2.3 Rigidification

There are forgetful functors

$$(Fib_{\text{scind}}/C) \xrightarrow{i} (Fib_{\text{cliv}}/C) \to (Fib/C).$$

The main goal of this section is to recall the fact that the first arrow, i, is an equivalence of categories. One constructs a specific rigidification functor

$$r : (Fib_{\text{cliv}}/C) \to (Fib_{\text{scind}}/C)$$

and shows that r is an inverse to i.

Let $F : T \to C$ be a fibered category with clivage $\phi^B_\phi : A^B_\phi \to A$. Define a new C-category $r(T)$ as follows. An object $\mathcal{A}$ of $r(T)_V$, for V in C, is a choice of an object $\mathcal{A}_W$ of T_W for every W in $C_{/V}$, together with fixed isomorphisms

$$\mathcal{A}(\phi) : \phi^*(\mathcal{A}_W) \xrightarrow{\simeq} \mathcal{A}_U$$

for $\phi : U \to W$ in $C_{/V}$. The $\mathcal{A}(\phi)$ should satisfy the condition that the diagrams

$$\begin{array}{ccc} \pi^*(\phi^*(\mathcal{A}_Z)) & \xrightarrow{\pi^*(\mathcal{A}(\phi))} & \pi^*(\mathcal{A}_W) \\ {\scriptstyle \lambda_{\phi,\pi}}\downarrow & & \downarrow{\scriptstyle \mathcal{A}(\pi)} \\ (\phi\circ\pi)^*(\mathcal{A}_Z) & \xrightarrow{\mathcal{A}(\phi\circ\pi)} & \mathcal{A}_U \end{array}$$

commute for all morphisms $\pi : U \to W$ and $\phi : W \to Z$ in $C_{/V}$. This defines the objects of $r(T)_V$ for all V. If $\mathcal{A}$ and $\mathcal{B}$ are objects of $r(T)_V$ define an element

$$f \in Hom_{r(T)}(\mathcal{A}, \mathcal{B})$$

to be a collection of morphisms

$$f_W : \mathcal{A}_W \to \mathcal{B}_W,$$

one for each object W of $C_{/V}$, such that the diagrams

$$\begin{array}{ccc} \phi^*(\mathcal{A}_W) & \xrightarrow{\mathcal{A}(\phi)} & \mathcal{A}_U \\ {\scriptstyle \phi^*(f_W)}\downarrow & & \downarrow{\scriptstyle f_U} \\ \phi^*(\mathcal{B}_W) & \xrightarrow{\mathcal{B}(\phi)} & \mathcal{B}_U \end{array}$$

commute for all $\phi : U \to W$ in $C_{/V}$.

Suppose that $\mathcal{A}$ is an object of $r(T)_V$, and let $\phi : W \to V$ be a morphism in C. Define $\phi^*(\mathcal{A})$ in $r(T)_W$ by setting

$$\phi^*(\mathcal{A})_U = \mathcal{A}_U$$

for U in $C_{/W} \subseteq C_{/V}$, and by setting

$$\phi^*(\mathcal{A})(\pi) = \mathcal{A}(\pi)$$

for π in $C_{/W}$.

If $\mathcal{A}$ is an object of $r(T)_V$, $\mathcal{B}$ is an object of $r(T)_W$, and $\phi : V \to W$ is a morphism in C, then let

$$Hom_\phi(\mathcal{A}, \mathcal{B}) = Hom_{r(T)}(\mathcal{A}, \phi^*(\mathcal{B})).$$

Lemma 2.3.1. *The construction $r(T)$ is a category.*

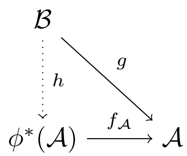
Lemma 2.3.2. *The functor $r(F) : r(T) \to C$ which sends an object of $r(T)_W$ to W and a morphism of $Hom_\phi(\mathcal{A}, \mathcal{B})$ to ϕ makes $r(T)$ into a C-category.*

Lemma 2.3.3. *Let $\phi : U \to V$ be a morphism in C. The morphism $f_{\mathcal{A}}$ of $Hom_\phi(\phi^*(\mathcal{A}), \mathcal{A})$ corresponding to*

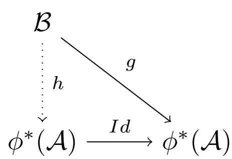
$$1 \in Hom_{r(T)}(\phi^*(\mathcal{A}), \phi^*(\mathcal{A})) = Hom_\phi(\phi^*(\mathcal{A}), \mathcal{A})$$

is a cartesian morphism over ϕ. In particular, $r(F) : r(T) \to C$ is pre-fibered.

Proof. Suppose that $g : \mathcal{B} \to \mathcal{A}$ is another morphism with $r(F)(g) = \phi$. Then, we want to fill in the dotted line so that the diagram



commutes. But, by definition, this diagram is the same as the diagram



in T_U. So, setting $h = g$ gives a unique solution to the problem. Thus, $f_{\mathcal{A}}$ is indeed cartesian. ☐

Lemma 2.3.4. *The pull-back morphisms $f_{\mathcal{A}}$ form a scindage.*

Lemma 2.3.5. *The C-category $r(T)$ is fibered. That is, compositions of cartesian morphisms are cartesian.*

Proof. This is an immediate corollary of Lemma 2.1.7. ☐

Lemma 2.3.6. *The construction $r : (Fib_{\mathrm{cliv}}/C) \to (Fib_{\mathrm{scind}}/C)$ is a functor.*

Lemma 2.3.7. *There is a natural cartesian functor*

$$i(r(T)) \to T$$

of fibered C-categories with clivage, and this functor is an equivalence of categories.

Proof. It is enough to check the equivalence on the fibers, where it is obvious. ☐

It is important to note that the rigidification functor r respects all relevant additional structure on a C-fibered category with clivage. In particular, it respects actions of groups and the presence of symmetric monoidal structure.

2.4 Stacks

Throughout the remainder of this chapter, C will denote a locally ringed Grothendieck site with terminal object U such that C is closed under finite fiber products. Therefore that the topology of the site C is given by a pre-topology, in the sense of [2, Definition II.1.3]. In this case, Čech cohomology of 1-hypercovers effectively computes $\mathrm{H}^2(U, A)$ for sheaves of abelian groups A [3, Theorem V.7.4.1], and these groups compute the group of A-gerbes [15, Theorem IV.3.4.2]. Recall that a 1-hypercover of U is a hypercover given by a cover $\mathcal{U}_I$ of covering morphisms indexed by I of U and a cover $\mathcal{V}_{ij}$ of each $U_{ij} = U_i \times_U U_j$. Assume for simplicity of notation that each $\mathcal{V}_{ij}$ is indexed by the set A. Then, the elements of $\mathcal{V}_{ij}$ will be written as V_{ij}^{α} for $\alpha \in A$. Denote such a hypercover by $\mathcal{V}_A \to \mathcal{U}_I \to U$, or more shortly by $\mathcal{U}^{\bullet} \to U$. The same notation will be used for covers, the 0-hypercovers.

Definition 2.4.1. Suppose that $F : T \to C$ is a fibered category with clivage. Suppose that $\mathcal{V}_A \to \mathcal{U}_I \to U$ is a hypercover. Then the hypercover gives rise to an augmented pseudo-cosimplicial category $T_U \to T_{\mathcal{U}^{\bullet}}$ in the usual way. The **descent category associated to $\mathcal{U}^{\bullet} \to U$** is defined to be

$$Des(\mathcal{U}^{\bullet} \to U) = \lim_{\Delta} T_{\mathcal{U}^{\bullet}}.$$

The augmentation determines a natural map

$$T_U \to Des(\mathcal{U}^\bullet \to U).$$

to the descent category.

Definition 2.4.2. A **stack** over a Grothendieck site C is a fibered category $F : T \to C$ with clivage such that the functors $T_W \to Des(\mathcal{U}^\bullet \to W)$ are equivalences of categories for all *covers* $\mathcal{U}^\bullet \to W$ in C.

Let $(Stacks/C)$ be the full subcategory of (Fib_{cliv}/C) with stacks as objects. Decorate $(Stacks/C)$ with symbols like $(G{-}Stacks/C)$ to denote stacks with an action of a presheaf (or sheaf) G of groups, or $(Stacks^{\otimes}/C)$ to denote stacks of symmetric monoidal categories, as above.

Definition 2.4.3. Call the objects of $(Stacks^{\otimes}/C)$ the **stacks of symmetric monoidal categories**.

Remark 2.4.4. The choice of clivage is not critical to the notion of a stack. Indeed, any two choices of clivage give rise to isomorphic pull-back functors, and hence to equivalent descent categories. So, if $T \to C$ is a stack with respect to some fixed clivage, it is a stack with respect to any other choice of clivage. In other words, the property of being a stack is really intrinsic to a fibered C-category.

2.5 Gluing stacks

Construction 2.5.1. Essentially by definition, one can glue stacks. It is worthwhile to detail concretely how this is done. Let C be a Grothendieck site with a terminal object U. If $V \to U$ is an object of C, then let C_V denote the induced site with terminal object V. Let $\mathcal{V}_A \to \mathcal{U}_I \to U$ be a 1-hypercover of U. Let $\alpha \in A$ index the objects of $\mathcal{V}_{ij}$, the cover of $U_i \times_U U_j$. So, V_{ij}^α will be a member of $\mathcal{V}_{ij}$. Suppose that $F_i : T_i \to C_{/U_i}$ are stacks. In order to descend to a stack on to C, one must first give equivalences of stacks

$$\sigma_{ij}^\alpha : T_j|V_{ij}^\alpha \to T_i|V_{ij}^\alpha,$$

for all $i, j \in I$, together with natural isomorphisms of functors

$$\gamma_{ijk}^{\alpha\beta\delta} : \sigma_{ij}^\alpha \circ \sigma_{jk}^\beta \Rightarrow \sigma_{ik}^\delta,$$

over

$$Z_{ijk}^{\alpha\beta\delta} = (V_{ij}^\alpha \times_{U_j} V_{jk}^\beta) \times_{U_k} V_{ik}^\delta$$

for all $i, j, k \in I$, all $\alpha, \beta, \delta \in A$. The γ are required to satisfy a cocycle condition: that the commutative diagrams induced by γ

$$\begin{array}{ccc} \sigma_{ij}^{\alpha} \circ \sigma_{jk}^{\beta} \circ \sigma_{kl}^{\delta} & \xrightarrow{\sigma_{ij}^{\alpha} \circ \gamma_{jkl}^{\beta\delta\epsilon}} & \sigma_{ij}^{\alpha} \circ \sigma_{jl}^{\epsilon} \\ {\scriptstyle \gamma_{ijk}^{\alpha\beta\eta} \circ \sigma_{kl}^{\delta}} \downarrow & & \downarrow {\scriptstyle \gamma_{ijl}^{\alpha\epsilon\tau}} \\ \sigma_{ik}^{\eta} \circ \sigma_{kl}^{\delta} & \xrightarrow{\gamma_{ikl}^{\eta\delta\tau}} & \sigma_{il}^{\tau} \end{array}$$

commute.

Now, for any object of C given by $\phi : W \to U$, define a descent category $T_W = Des(W \times_U \mathcal{U}^\bullet \to W)$. The idea is then that these descent categories define the stack globally on C. An object of T_W consists of objects A_i of $(T_i)_{W \times_U U_i}$ for all $i \in I$, together with isomorphisms

$$\beta_{ij}^{\alpha} : \sigma_{ij}^{\alpha}(A_j|_{V_{ij}^{\alpha}}) \to A_i|_{V_{ij}^{\alpha}},$$

such that the diagram

$$\begin{array}{ccc} \sigma_{ij}^{\alpha}(\sigma_{jk}^{\delta}(A_k|_{Z_{ijk}^{\alpha\delta\epsilon}})) & \xrightarrow{\sigma_{ij}^{\alpha}(\beta_{jk}^{\delta})} & \sigma_{ij}^{\alpha}(A_j|_{Z_{ijk}^{\alpha\delta\epsilon}}) \\ {\scriptstyle \gamma_{ijk}^{\alpha\delta\epsilon}} \downarrow & & \downarrow {\scriptstyle \beta_{ij}^{\alpha}} \\ \sigma_{ik}^{\epsilon}(A_k|_{Z_{ijk}^{\alpha\delta\epsilon}}) & \xrightarrow{\beta_{ik}^{\epsilon}} & A_i|_{Z_{ijk}^{\alpha\delta\epsilon}} \end{array}$$

is commutative.

Morphisms between objects of $T_W = Des(W \times_U \mathcal{U}^\bullet \to W)$ are given by a collection of morphisms $A_i \to B_i$ that commute with the morphisms β_{ij}^{α}. If $\pi : V \to W$ is a morphism in C, then there are natural functors

$$\pi^* : T_W = Des(W \times_U \mathcal{U}^\bullet \to W) \to T_V = Des(V \times_U \mathcal{U}^\bullet \to V)$$

defined by clivage in each of the stacks T_i.

Finally, if $\pi : V \to W$ is an arrow in C, and if A is an object of $T_V = Des(V \times_U \mathcal{U}^\bullet \to V)$ and B is an object of $T_W = Des(W \times_U \mathcal{U}^\bullet \to W)$, then define

$$Hom_T(A, B) = Hom_{T_V}(A, \pi^*(B)).$$

This defines a new C-category T with clivage.

Lemma 2.5.2. *The category whose objects are descent data as defined above for all objects $\phi : W \to U$ in C defines a stack over C. Moreover, there are natural equivalences of $C_{/U_i}$-stacks*

$$T_i \to T_{/U_i}$$

Proof. First, there are natural morphisms

$$T_{i,U_i} \to Des(U_i \times_U \mathcal{U}^\bullet \to U_i) \to Des(U_i \times_U \mathcal{U}^\bullet \to U_i),$$

where the first descent category is in the stack T_i and the second is in the fibered category C. These are equivalences, so it follows that $T_{/U_i}$ is in fact a stack, equivalent to T_i. To show that T is also a stack, it suffices to show that

$$T_U \to Des(\mathcal{W}^\bullet \to U)$$

is an equivalence for any cover $\mathcal{W}$ of U. But, this is true because the descent categories are defined via objects over $W_j \times_U U_i$, where one knows descent holds. □

Definition 2.5.3. Suppose that $\Sigma = (S_i, \sigma_{ij}^\alpha, \gamma_{ijk}^{\alpha\beta\gamma})$ and $\Psi = (T_i, \sigma_{ij}^\alpha, \gamma_{ijk}^{\alpha\beta\gamma})$ are two sets of gluing data over a 1-hypercover $\mathcal{V} \to \mathcal{U} \to U$. A morphism $\Phi : \Sigma \to \Psi$ consists of morphisms of stacks

$$\Phi_i : S_i \to T_i$$

such that the diagrams

$$\begin{array}{ccc} S_i & \xrightarrow{\Phi_i} & T_i \\ {\scriptstyle\sigma_{ij}^\alpha}\downarrow & & \downarrow{\scriptstyle\sigma_{ij}^\alpha} \\ S_j & \xrightarrow{\Phi_j} & T_j \end{array}$$

are commutative and $\Phi_i(\gamma_{ijk}^{\alpha\beta\gamma}) = \gamma_{ijk}^{\alpha\beta\gamma}(\Phi_i)$.

Lemma 2.5.4. *A morphism $\Phi : \Sigma \to \Psi$ as above defines a morphism $\Phi : S \to T$ of stacks over C. Moreover, for all U_i in $\mathcal{U}$, the diagrams*

$$\begin{array}{ccc} S_i & \longrightarrow & S_{/U_i} \\ {\scriptstyle\Phi_i}\downarrow & & \downarrow{\scriptstyle\Phi_{/U_i}} \\ T_i & \longrightarrow & T_{/U_i} \end{array}$$

are commutative.

2.6 Gerbes and the Cohomological Brauer Group

If A is a sheaf of groups on a site C, then one defines a stack of A-torsors $Tors(A)$. The fiber $Tors(A)_V$ consists of $A|_V$-torsors on V. A map of A-torsors $a : A \to B$ that lies over a morphism $\phi : V \to W$ is an isomorphism $A \xrightarrow{\simeq} \phi^*(B)$. Write **Pic** for the stack of $\mathbb{G}_m$-torsors. In fact, these torsor stacks are gerbes.

Definition 2.6.1. A gerbe over a Grothendieck site C is a stack G satisfying three conditions: the fiber categories must all be groupoids, there exists a cover $\mathcal{U}^\bullet \to U$ such that each G_{U_i} is non-empty, and for any two objects $A, B \in G_W$, there exists a cover $\mathcal{V}^\bullet \to W$ such that there are isomorphisms $\phi_i^*(A) \xrightarrow{\cong} \phi_i^*(B)$ in each G_{V_i}.

This definition may be summed up by saying that a gerbe is a stack whose fibers are groupoids such that the stalks are connected.

Definition 2.6.2. Let A be a sheaf of abelian groups on C. Any gerbe G locally equivalent to $Tors(A)$ is called an A-gerbe. Here, local equivalence means that there is a covering morphism $\mathcal{U}^\bullet \to U$, and there are equivalences of stacks $\phi_i^*(G) \to \phi_i^*(Tors(A))$ for all i.

Proposition 2.6.3. *Let A be a sheaf of abelian groups in the étale topology on U. Then, equivalence classes of A-gerbes are classified by the cohomology group* $\mathrm{H}^2(U_{\text{ét}}, A)$.

Proof. I only sketch the proof. For details, see [15, Theorem IV.3.4] or [7, Theorem 5.2.8] This sketch is applicable for U quasi-separated, where the étale site has fiber products and finite products. In this case, sheaf cohomology is computable with cocycles in hypercovers [3, Theorem V.7.4.1]. To say that a gerbe G is an A-gerbe is to say that there is a cover $\mathcal{U}_I$ of U, there are objects $a_i \in G_{U_i}$, and there exist isomorphisms $\sigma_i : \mathrm{Aut}(a_i) \xrightarrow{\cong} A|_{U_i}$. Indeed, in this case, if $b \in G_{U_i}$, then $\mathrm{Iso}(a_i, b)$ is a $\mathrm{Aut}(a_i)$-torsor, and hence, via σ_i^{-1}, a $A|_{U_i}$-torsor. Together, the a_i and σ_i give an equivalence of gerbes $G|_{U_i} \to Tors(A)|_{U_i}$. Showing that it is actually an equivalence simply amounts to using descent. Indeed, if $\mathrm{Iso}(a_i, b)$ is the trivial A-torsor, then there is an isomorphism $a_i \to b$ over U_i. On the other hand, if L is an A-torsor over U_i, then one can take a cover on which it is trivial, and use the gluing datum to create a descent data for a_i. This yields an object b_L of G_{U_i} with $\mathrm{Iso}(a_i, b_L)$ isomorphic to L.

Recall how to associate an element of $\mathrm{H}^2(U, A)$ to an A-gerbe G. Let $\mathcal{U}_I$ as above be a cover of U that trivializes G. Let, for each $i, j \in I$, $\mathcal{V}_{ij}$ be a cover of $U_{ij} = U_i \times_U U_j$ such that on each V_{ij}^α there is a morphism $\theta_{ij}^\alpha : a_i|_{V_{ij}^\alpha} \to a_j|_{V_{ij}^\beta}$. Set $Z_{ijk}^{\alpha\beta\gamma} = V_{ij}^\alpha \times_U V_{ik}^\gamma \times_U V_{jk}^\beta$. Then,

$$\sigma_i((\theta_{ik}^\gamma)^{-1}|_{Z_{ijk}^{\alpha\beta\gamma}} \circ \theta_{jk}^\beta|_{Z_{ijk}^{\alpha\beta\gamma}} \circ \theta_{ij}^\alpha|_{Z_{ijk}^{\alpha\beta\gamma}})$$

gives an element of $A(Z_{ijk}^{\alpha\beta\gamma})$. It is not hard to check that this gives a 2-cocycle for the hypercover $\mathcal{V}_A \to \mathcal{U}_I \to U$. And, the cocycle in

$$\mathrm{H}^2(U, A) \cong \check{\mathrm{H}}^2(U, A) = \mathrm{colim}_{1-\text{hypercovers}}\, \check{\mathrm{H}}^2(\mathcal{U}^\bullet, A)$$

is well-defined and depends only on the gerbe G up to equivalence of cocycles. The next construction gives the inverse. □

Construction 2.6.4. For the notation, see Construction 2.5.1. Fix $\alpha \in \mathrm{H}^2(U, A)$. Let α be determined by a class $\alpha_{ijk}^{\alpha\beta\delta} \in \check{\mathrm{H}}^2(\mathcal{U}^\bullet, A)$ for a 1-hypercover $\mathcal{V}_A \to \mathcal{U}_I \to U$. Then, on each U_i in $\mathcal{U}_I$, let $G_i = Tors(A)|_{U_i}$. On the overlaps $U_i \times_U U_j$, set

$$\sigma_{ij} = Id : p_2^*(G_j) \xrightarrow{=} p_1^*(G_i).$$

Thus, the overlap maps are all the identity. Let $\gamma_{ijk}^{\alpha\beta\delta}$ be multiplication by $\alpha_{ijk}^{\alpha\beta\delta}$, as a natural transformation of the identity on the category of A-torsors. The cocycle condition for $\gamma_{ijk}^{\alpha\beta\delta}$ follows from the cocycle condition for $\alpha_{ijk}^{\alpha\beta\delta}$. The corresponding gerbe determined by this gluing data is denoted by $Tors(A)^\alpha$. Write $\mathbf{Pic}^\alpha$ for $Tors(\mathbb{G}_m)^\alpha$ when $\alpha \in \mathrm{H}^2(U, \mathbb{G}_m)$.

2.7 Twisting Functors

Let A be a sheaf of abelian groups on C, let $\mathcal{V} \to \mathcal{U} \to U$ be a 1-hypercover, and let $\check{\alpha} \in \check{Z}^2(\mathcal{V}, A)$ be a 2-cocycle. The Construction 2.6.4 defines a functor

$$\check{\alpha} : (A\text{–}Stacks/C) \to (Stacks/C),$$

called the $\check{\alpha}$-twisting functor. Denote by $(A\text{–}Stacks/C)^{\check{\alpha}}$ the essential image category of the functor. If $\mathcal{W} \to \mathcal{Z} \to U$ is another 1-hypercover, and if $\check{\beta}$ is the image of $\check{\alpha}$ in $\check{Z}^2(\mathcal{W}, A)$, then $\check{\alpha}$ factors through $\check{\beta}$ as functors

$$(A\text{–}Stacks/C) \to (Stacks/C).$$

Similarly, if $\check{\alpha} = \check{\beta}$ in $\check{\mathrm{H}}^2(\mathcal{V}, A)$, then the corresponding twisting functors are canonically equivalent. This establishes the following proposition.

Proposition 2.7.1. *For any $\alpha \in \mathrm{H}^2(U, A)$, there is a naturally defined twisting functor*

$$\alpha : (A\text{–}Stacks/C) \to (Stacks/C).$$

determined by the Čech twisting functors $\check{\alpha}$ above.

In general, if T is an A-stack, then I will write T^α for $\alpha(T)$.

Remark 2.7.2. Everything in this section works equally well to create twisting functors

$$\alpha : (A\text{–}Stacks^{\otimes}/C) \to (Stacks^{\otimes}/C).$$

In particular, there are natural categories $(A\text{–}Stacks^{\otimes}/C)^\alpha$.

Now, let C be the étale topology on some geometrically connected quasi-separated scheme U, and let $\beta \in \mathrm{H}^2(U_{\text{ét}}, \mu_n)$, where n is invertible in all residue fields of all points of U. Let $\beta \mapsto \alpha$ in $\mathrm{H}^2(U_{\text{ét}}, \mu_n) \to \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$. The stack of symmetric monoidal categories $\mathbf{Proj}$, consisting of locally project coherent $\mathcal{O}_U$-modules of finite rank, is naturally an object of both $(\mathbb{G}_m\text{–}Stacks^{\otimes}/C)$ and $(\mu_n\text{–}Stacks^{\otimes}/C)$. Moreover, the stacks of symmetric monoidal categories $\mathbf{Proj}^\beta = \beta(\mathbf{Proj})$ and $\mathbf{Proj}^\alpha = \alpha(\mathbf{Proj})$ are monoidally equivalent.

Let $\mathbf{nSets}$ be the stack of symmetric monoidal categories consisting of disjoint unions of μ_n-torsors. This is the stack of sheaves of finite and faithful μ_n-sets. There is a natural monoidal μ_n-cartesian functor

$$u : \mathbf{nSets} \to \mathbf{Proj}$$

that takes a μ_n-torsor to the corresponding $\mathbb{G}_m$-torsor and then to associated projective coherent $\mathcal{O}_U$-module of rank one and takes disjoint unions to direct sums. This is essentially the unit morphism in the category of μ_n-$\otimes$-stacks on which μ_n acts faithfully in an appropriate sense.

Definition 2.7.3. For $\beta \in \mathrm{H}^2(U_{\text{ét}}, \mu_n)$, as above, the **twisted unit morphism** is the functor

$$\beta\left(\mathbf{nSets} \to \mathbf{Proj}\right) = \mathbf{nSets}^\beta \xrightarrow{\beta(u)} \mathbf{Proj}^\beta.$$

in $(\mu_n\text{–}Stacks^{\otimes}/C)^\beta$.

The motivation for this definition comes from the utility of the unit map $S \to K(\mathbb{Z})$, where S is the sphere spectrum. The sphere spectrum may be seen as the K-theory of the free symmetric monoidal category $*$ on one object. There is a natural map $* \to \mathbf{Proj}$ that takes n to the free $\mathcal{O}_U$-module on n elements, but it is not a map of μ_n-sets so that it doesn't give a unit map to $\mathbf{Proj}^\alpha$.

Remark 2.7.4. It is not obvious at first whether there should in general exist non-trivial global α-twisted locally free and finite rank sheaves in $\mathbf{Proj}_U^\alpha$ for $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$. In fact, this is equivalent to the question of whether α is representable by an Azumaya algebra $\mathcal{A}$. Indeed, given a non-trivial α-twisted finite rank projective sheaf $\mathcal{E}$, the endomorphism sheaf $End(\mathcal{E})$ is an Azumaya algebra representing α. In the other direction, one uses the fact that $\mathcal{A}$ is locally a matrix algebra over $\mathcal{O}_U$. For details, I again refer the reader to [10].

2.8 K-Theory

Definition 2.8.1. Take as K-theory functor the level one part of the functor Spt from symmetric monoidal categories to spectra of [27, Section 1.6]. Thus, $\mathbf{K}$ is a functor

$$\mathbf{K} : SymMon \to \mathbf{sSets},$$

from the category of symmetric monoidal categories and lax functors to simplicial sets.

Definition 2.8.2. The **K-theory presheaf associated to a category fibered in symmetric monoidal categories** is defined to be the composition

$$V \mapsto \mathbf{K}(r(T)_V),$$

where r is the rigidification functor.

This defines a functor

$$\mathbf{K} : (Fib^{\otimes}_{\text{cliv}}/C) \to \mathbf{sPre}(C),$$

from C-categories fibered in symmetric monoidal categories to simplicial presheaves on C.

The level zero space will not work for the applications below, because, in the version of the Brown-Gersten spectral sequence for presheaves of simplicial sets, all differentials emerging from $\mathrm{H}^0(U, \pi_0 X)$ are identically zero. Therefore, if T is a symmetric monoidal category, then $\pi_k(\mathbf{K}(T)) = K_{k-1}(T)$ for $k \geq 1$.

2.9 K-Theory of Monomial Matrices

Definition 2.9.1. For $\beta \in \mathrm{H}^2(U_{\text{ét}}, \mu_n)$, I will let $\mathbf{T}^\beta$ denote the presheaf

$$\mathbf{T}^\beta = \mathbf{K}(\mathbf{nSets}^\beta).$$

Define

$$\mathbf{T}^\beta_k(V) = \pi_{k+1}\mathbf{T}^\beta(V),$$

and let $\mathcal{T}^\beta_k$ be the sheafification of $\mathbf{T}^\beta_k$.

Definition 2.9.2. Similarly, for $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$, let $\mathbf{K}^\alpha$ be the presheaf $\mathbf{K}(\mathbf{Proj}^\alpha)$, with associated homotopy presheaves

$$\mathbf{K}^\alpha_k(V) = \pi_{k+1}\mathbf{T}^\alpha(V),$$

and presheaves $\mathcal{K}^\alpha_k$.

If $\beta \mapsto \alpha$ in $\mathrm{H}^2(U_{\text{ét}}, \mu_n) \to \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$, then the twisted unit morphism gives a map of presheaves

$$\mathbf{T}^\beta \to \mathbf{K}^\alpha.$$

This map will be crucial to the proof of the period-spectral index theorem.

Every μ_n-set is a disjoint union of μ_n-torsors. The stalk of the stack $\mathbf{nSets}$ at a geometric point $\overline{x} \to U$ is therefore equivalent to

$$\coprod_{j \geq 0} S_j \wr \mu_n(k(\overline{x})),$$

where S_j is the symmetric group on j letters, and $S_j \wr \mu_n$ is the wreath product. This notation means that the stalk is equivalent to the groupoid with connected components indexed by $j \geq 0$, where the automorphism group of an object in the jth component is

$$S_j \wr \mu_n(k(\overline{x})).$$

This is true in the étale topology as the local ring of a geometric point is Henselian. In other words, the stalk

$$\mathbf{nSets}_{\overline{x}}$$

is the free symmetric monoidal category on the groupoid $\mu_n(k(\overline{x}))$. Therefore, by Thomason [28, Lemma 2.5], the K-theory spectrum of this symmetric monoidal category is weak equivalent to the suspension spectrum $\Sigma^\infty(B\mu_n(k(\overline{x})))_+$ of the classifying space of $B\mu_n((k\overline{x}))$ with a disjoint basepoint. This spectrum is weak equivalent to $\Sigma^\infty\left(B\mu_n(k(\overline{x})) \vee S^0\right)$. When n is prime to the characteristic of $k(\overline{x})$, the stable homotopy is

$$K_j(\mathbf{nSets}_{\overline{x}}) \xrightarrow{\simeq} \pi_j^s((B\mu_n(k(\overline{x})))_+) \xrightarrow{\simeq} \pi_j^s(B\mu_n(k(\overline{x}))) \oplus \pi_j^s \cong \pi_j^s(B\mathbb{Z}/(n)) \oplus \pi_j^s,$$

where $\pi_j^s = \pi_k^s(S^0)$. Note that

$$K_j(\mathbf{nSets}_{\overline{x}}) \cong \pi_{j+1}\mathbf{T}_{\overline{x}},$$

since K-theory commutes with filtered colimits.

2.10 Homotopy Sheaves are Isomorphic

Proposition 2.10.1. *Fix an element $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$. Then, for all $n \geq 0$, the homotopy sheaves $\pi_n(\mathbf{K}^\alpha)$ and $\pi_n(\mathbf{K})$ are naturally isomorphic. Similarly, if $\beta \in \mathrm{H}^2(U_{\text{ét}}, \mu_n)$, then $\pi_n(\mathbf{T}^\beta) \cong \pi_n(\mathbf{T})$.*

Proof. Here is a proof for the case of $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$. The proof of the other case is identical.

Let $\mathcal{U}_I \to U$ be a cover over which α is trivial. Then, the gerbe $\mathbf{Pic}^\alpha$ is trivial on $\mathcal{U}_I$. Thus, there exist α-twisted line bundles $\mathcal{L}_i$ on each U_i. These define equivalences $\theta_i : \mathbf{Proj}|_{U_i} \to \mathbf{Proj}^\alpha|_{U_i}$ for all i given by

$$\theta_i(V)(\mathcal{P}) = \mathcal{L}_i \otimes \mathcal{P},$$

when $V \to U_i$. These equivalences induce point-wise weak equivalences of K-theory presheaves: $\theta_i : \mathbf{K}|_{U_i} \to \mathbf{K}^\alpha|_{U_i}$. It follows that on U_i there are isomorphisms of homotopy presheaves:

$$\theta_i : \pi_n^p(\mathbf{K})|_{U_i} \to \pi_n^p(\mathbf{K}^\alpha)|_{U_i}.$$

In fact, the θ_i glue at the level of homotopy sheaves. Since in the cover I might have $U_i = U_j$, and I can take different line bundles $\mathcal{L}_i$ and $\mathcal{L}_j$, this will imply that the resulting morphisms on homotopy sheaves of K-theory are independent of the choice of the line bundles $\mathcal{L}_i$. It will also show that the morphisms do not depend on the cover $\mathcal{U}_I$.

To show that the θ_i glue, it suffices to check that, on $U_{ij} = U_i \times_U U_j$, the auto-equivalence of $\mathbf{Proj}|_{U_{ij}}$ given by tensoring by $\mathcal{M}_{ij} = \mathcal{L}_i^{-1} \otimes \mathcal{L}_j$ is locally homotopic to the identity. But, I can take a trivialization of $\mathcal{M}_{ij}$, over a cover $\mathcal{V}$ of U_{ij}. So, on each element V of $\mathcal{V}$, there is an isomorphism $\sigma_V : \mathcal{O}_{U_V} \xrightarrow{\simeq} \mathcal{M}_{ij}|_V$. This induces a natural transformation from the identity to $\theta_i^{-1} \circ \theta_j$ on V. So, on V, I see that $\theta_i|_V = \theta_j|_V : \pi_n^p(\mathbf{K})|_V \to \pi_n^p(\mathbf{K}^\alpha)|_V$. It follows that the θ_i glue to give isomorphisms *of sheaves*

$$\theta : \pi_n(\mathbf{K}) \to \pi_n(\mathbf{K}^\alpha),$$

as desired. □

2.11 Intrinsic Properties of Fibered Categories

This section is dedicated to explaining how to make the definitions of a stack, Definition 2.4.2, and a category fibered in symmetric monoidal categories, Definition 2.2.1, truly intrinsic. David Gepner explained to me how symmetric monoidal categories are the same as categories cofibered over Γ with coclivage satisfying the Segal condition.

Let $F : T \to C$ be a fibered category, where C is a Grothendieck site. Let U be an object of C, and let $\mathcal{U}_I \to U$ be a covering of U.

Definition 2.11.1. A **gluing datum** (A, B, λ, C) consists of the following data:

1. an object A_i of T_{U_i} for each $i \in I$;

2. cartesian morphisms $B_{ij}^1 \to A_i$ and $B_{ij}^2 \to A_j$ over $U_{ij} \to U_i$ and $U_{ij} \to U_j$, respectively;

3. isomorphisms

$$\lambda_{ij} : B_{ij}^2 \to B_{ij}^1;$$

4. and cartesian morphisms

$$C_{ijk}^{1,12} \to B_{ij}^{1} \to A_i$$
$$C_{ijk}^{1,13} \to B_{ik}^{1} \to A_i$$
$$C_{ijk}^{1,23} \to B_{jk}^{1} \to A_j$$
$$C_{ijk}^{2,12} \to B_{ij}^{2} \to A_j$$
$$C_{ijk}^{2,13} \to B_{ik}^{2} \to A_k$$
$$C_{ijk}^{2,23} \to B_{jk}^{2} \to A_k$$

over the appropriate morphisms in C.

Lemma 2.11.2. *Given a gluing datum (A, B, λ, C), there are uniquely defined isomorphisms*

$$\theta_i : C_{ijk}^{1,12} \xrightarrow{\simeq} C_{ijk}^{1,13}$$
$$\theta_j : C_{ijk}^{1,23} \xrightarrow{\simeq} C_{ijk}^{2,12}$$
$$\theta_k : C_{ijk}^{2,13} \xrightarrow{\simeq} C_{ijk}^{2,23}$$

by the cartesian property. Similarly, the λ_{ij} define unique isomorphisms

$$\lambda_{ij} : C_{ijk}^{2,12} \to C_{ijk}^{1,12}$$
$$\lambda_{jk} : C_{ijk}^{2,23} \to C_{ijk}^{1,23}$$
$$\lambda_{ik} : C_{ijk}^{2,13} \to C_{ijk}^{1,13}.$$

Definition 2.11.3. If a gluing datum (A, B, λ, C) is such that the diagrams

$$\begin{array}{ccccc}
C_{ijk}^{2,13} & \xrightarrow{\theta_k} & C_{ijk}^{2,23} & \xrightarrow{\lambda_{jk}} & C_{ijk}^{1,23} \\
 & & & & \downarrow \theta_j^{-1} \\
\downarrow \lambda_{ik} & & & & C_{ijk}^{2,12} \\
 & & & & \downarrow \lambda_{ij} \\
C_{ijk}^{1,13} & & \xleftarrow[\theta_i^{-1}]{} & & C_{ijk}^{1,12}
\end{array}$$

are commutative, then it is called a **gluing object**.

Definition 2.11.4. Let C be a Grothendieck site, and let $\mathcal{U}_I \to U$ be a covering in C. Define $\overline{Des}(\mathcal{U} \to U)$ to be the category whose objects are the gluing objects. A morphism of gluing objects $({}_IA, {}_IB, {}_I\lambda, {}_IC) \to ({}_{II}A, {}_{II}B, {}_{II}\lambda, {}_{II}C)$ is a collection of morphisms $f_i : {}_IA_i \to {}_{II}A_i$ of T_{U_i} such that the middle square, which is induced by the cartesian property of the left and right-hand horizontal morphisms, of

$$\begin{array}{ccccccc} {}_IA_j & \longleftarrow & {}_IB^2_{ij} & \xrightarrow{{}_I\lambda_{ij}} & {}_IB^1_{ij} & \longrightarrow & {}_IA_i \\ \downarrow f_j & & \downarrow & & \downarrow & & \downarrow f_i \\ {}_{II}A_j & \longleftarrow & {}_{II}B^2_{ij} & \xrightarrow{{}_{II}\lambda_{ij}} & {}_{II}B^1_{ij} & \longrightarrow & {}_{II}A_i \end{array}$$

is commutative.

Definition 2.11.5. An **effective gluing object** for $\mathcal{U} \to U$ is a collection of data as follows:

1. an object A of T_U;
2. cartesian morphisms $A_i \to A$ over $U_i \to U$ for all $i \in I$;
3. cartesian morphisms B^1_{ij} and B^2_{ij}, as above;
4. cartesian morphisms $C^{1,12}_{ijk}$ and so on, as above.

A morphism of effective gluing objects is simply given by a morphism on the objects over T_U. Call the category of such objects and morphisms $\overline{Eff}(\mathcal{U} \to U)$.

Given an effective gluing object, since the compositions $B^2_{ij} \to A_j \to A$ and $B^1_{ij} \to A_i \to A$ lie over the same morphism of C, there exists a unique isomorphism

$$\lambda_{ij} : B^2_{ij} \to B^1_{ij}.$$

In a natural way, this determines a functor

$$\overline{Eff}(\mathcal{U} \to U) \to \overline{Des}(\mathcal{U} \to U)$$

given by forgetting A_U.

Definition 2.11.6. The **intrinsic stack condition** is the condition that

$$\overline{Eff}(\mathcal{V} \to V) \to \overline{Des}(\mathcal{V} \to V)$$

is an equivalence for all covers $\mathcal{V} \to V$ in C.

Proposition 2.11.7. *Let $F : T \to C$ be a fibered category. Then, T is a stack (with respect to one and hence any choice of clivage) if and only if T satisfies the intrinsic stack condition.*

Proof. There is a commutative diagram

$$\begin{array}{ccc} T_V & \longrightarrow & Des(\mathcal{V} \to V) \\ \downarrow & & \downarrow \\ \overline{Eff}(\mathcal{V} \to V) & \longrightarrow & \overline{Des}(\mathcal{V} \to V), \end{array}$$

where the vertical morphisms are defined by some fixed clivage for T. But, the vertical arrows are clearly equivalences. Thus, the top arrow is an equivalence if and only if the bottom arrow is. □

This proposition provides an intrinsic characterization of those fibered categories which are stacks over C. Now, I turn to the problem of giving an intrinsic notion of a category fibered in symmetric monoidal categories. Note that the notions of cocartesian morphisms, cofibered categories, coclivage, etc. are obtained by taking a fibered category $T \to C$ and looking at $T^{\mathrm{op}} \to C^{\mathrm{op}}$.

Let Γ be the category of finite pointed sets and base-point preserving functions. Let $[n] = \{0, 1, \ldots, n\}$ be pointed by 0. One somewhat intrinsic notion of a symmetric monoidal category is a category $F : T \to \Gamma$, cofibered over Γ, together with coclivage such that

$$(i_1)_* \times \cdots \times (i_n)_* : T_{[n]} \to (T_{[1]})^n$$

is an equivalence of categories for all n, where $i_k : [n] \to [1]$ sends k to 1 and all other elements to the base-point of $[1]$. This is called the **Segal condition**.

Definition 2.11.8. Let $F : T \to \Gamma$ be a cofibered category with coclivage. Say that T is a **symmetric monoidal category** if the Segal condition holds.

Let $\overline{T}_{[n]}$ for $n > 1$ be the category consisting of objects an object A of $T_{[n]}$ together with a cocartesian morphism $A \to A_k$ over $i_k : [n] \to [1]$. The morphisms are clear, and there is a natural functor $\overline{T}_{[n]} \to (T_{[1]})^n$.

Definition 2.11.9. The **intrinsic Segal condition** is the condition that for all $n > 1$,

$$\overline{T}_{[n]} \to (T_{[1]})^n$$

is an equivalence.

Proposition 2.11.10. *Let $F : T \to \Gamma$ be a cofibered category. Then, T is a symmetric monoidal category (with respect to one and hence any choice of coclivage) if and only if T satisfies the intrinsic Segal condition.*

Proof. It suffices to note that if we have fixed a coclivage for T, then there is a natural equivalence of categories

$$T_{[n]} \to \overline{T}_{[n]}$$

and that the functor $T_{[n]} \to (T_{[1]})^n$ of the Segal condition factors through this. $\square$

This gives an intrinsic definition of a category to be symmetric monoidal. Finally, I will explain the notion of a category fibered in symmetric monoidal categories.

Definition 2.11.11. Let $C \times D$ be a product category. A C-morphism is a morphism $(U, u) \to (V, u)$. Similarly, a D-morphism is a morphism $(U, u) \to (U, v)$. Let $F : T \to C \times D$. A C-cartesian morphism is a cartesian morphism over a C-morphism. A D-cocartesian morphism is a cocartesian morphism over a D-morphism.

Definition 2.11.12. For the purposes of this section, a category $F : T \to C \times D$ is called **cross-fibered** if C-cartesian morphisms exist over every C-morphism, if compositions of C-cartesian morphisms are C-cartesian, if D-cocartesian morphisms exist over every D-morphism, and if compositions of D-cocartesian morphisms are D-cocartesian.

For an object u of D, let $T_{(-,u)}$ be $T \times_{C\times D} (C \times \{u\})$. Similarly, for U in C, let $T_{(U,-)}$ be $T \times_{C\times D} (\{U\} \times D)$.

Lemma 2.11.13. *If $F : T \to C \times D$ is cross-fibered, then $T_{(-,u)} \to C$ is fibered for all u in D, and $T_{(U,-)} \to D$ is cofibered for all U in C.*

Definition 2.11.14. A category fibered in symmetric monoidal categories is a cross-fibered category over $C \times \Gamma$ such that each $T_{(U,-)}$ satisfies the intrinsic Segal conditions.

Definition 2.11.15. A **stack of symmetric monoidal categories** is a cross-fibered category over $C \times \Gamma$ satisfying the intrinsic stack conditions on $T_{(-,[1])}$ and the intrinsic Segal conditions on each $T_{(U,-)}$.

Proposition 2.11.16. *If $F : T \to C \times \Gamma$ is a category fibered in symmetric monoidal categories, then, for any choice of coclivage,*

$$T_{(-,[n])} \to \left(T_{(-,[1])}\right)^n$$

is an equivalence of fibered categories for all $n > 0$.

Corollary 2.11.17. *For a stack of symmetric monoidal categories $T \to C \times \Gamma$, each $T_{(-,[n])}$ is a stack over C.*

Chapter 3

The Spectral Index

In this chapter, the spectral index is defined, and the **obstruction** and **bound** properties of the introduction are proven. In Section 3.1, some preliminary computations are made which will figure in the application of the **bound** property to Brauer classes on fields, to obtain the period-spectral index theorem, Theorem 3.3.8. In Section 3.2, the **obstruction** property is established. In Section 3.3, the spectral index is defined, and the **bound** property is proven.

3.1 Stable Homotopy of Classifying Spaces

Proposition 3.1.1. *Let $0 < k < 2p-3$. Then, the p-primary component $\pi_k^s(p)$ of π_k^s is zero. And,*

$$\pi_{2p-3}^s(p) = \mathbb{Z}/(p).$$

Proof. This follows from the computation of the image of the J-morphism (see [24, Theorem 1.1.13]) and, for example, [24, Theorem 1.1.14]. □

I thank Peter Bousfield for telling me about the next proposition.

Proposition 3.1.2. *Let $G = \mathbb{Z}/(p^n)$. Then, for $0 < k < 2p-2$, the stable homotopy group $\pi_k^s(BG)$ is isomorphic to $\mathbb{Z}/(p^n)$ for k odd and 0 for k even.*

Proof. Let p be a prime. Recall the stable splitting of Holzsager [18]

$$\Sigma B\mathbb{Z}/(p^n) \xrightarrow{\simeq} X_1 \vee \cdots \vee X_{p-1},$$

where, if $k > 0$, the reduced homology of X_m is

$$\tilde{\mathrm{H}}_k(X_m, \mathbb{Z}) \xrightarrow{\simeq} \begin{cases} \mathbb{Z}/(p^n) & \text{if } k \cong 2m \mod 2p-2, \\ 0 & \text{otherwise.} \end{cases}$$

Define C_m as the cofiber of

$$M(\mathbb{Z}/(p^n), 2m) \to X_m,$$

where $M_1 = M(\mathbb{Z}/(p^n), 2m)$ is the Moore space with

$$\tilde{\mathrm{H}}_k(M_1, \mathbb{Z}) \xrightarrow{\cong} \begin{cases} \mathbb{Z}/(p^n) & \text{if } k = 2m, \\ 0 & \text{otherwise,} \end{cases}$$

when $k > 0$.

The homology of C_m is

$$\tilde{\mathrm{H}}_k(C_m, \mathbb{Z}) \xrightarrow{\cong} \begin{cases} \mathbb{Z}/(p^n) & \text{if } k > 2m \text{ and } k \cong 2m \mod 2p-2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the map

$$M_2 = M(\mathbb{Z}/(p^n), 2m + 2p - 2) \to C_m$$

is a $(2m + 4p - 5)$-equivalence. Thus, for $k < 2m + 4p - 5$ (resp. $k = 2m + 4p - 5$), the map

$$\pi_k^s(M_2) \to \pi_k^s(C_m)$$

is an isomorphism (resp. surjection). Therefore, there is an exact sequence

$$\begin{aligned} \pi^s_{2m+4p-5}(M_2) \to \pi^s_{2m+4p-6}(M_1) \to \pi^s_{2m+4p-6}(X_m) \to \pi^s_{2m+4p-6}(M_2) \to \cdots \\ \to \pi^s_k(M_1) \to \pi^s_k(X_m) \to \pi^s_k(M_2) \to \cdots \end{aligned} \tag{3.1}$$

Let $M(\mathbb{Z}/(p^n))$ be the Moore spectrum. It is the cofiber of the multiplication by p^n map on the sphere spectrum S. Thus, its stable homotopy groups fit into exact sequences

$$0 \to \pi_k^s \otimes_{\mathbb{Z}} \mathbb{Z}/(p^n) \to \pi_k(M(\mathbb{Z}/(p^n))) \to \mathrm{Tor}_1^{\mathbb{Z}}(\pi_{k-1}^s, \mathbb{Z}/(p^n)) \to 0.$$

These sequences are in fact split when p is odd or when $p = 2$ and $n > 1$. The Moore spaces M_1 and M_2 are the level $2m$ and $(2m + 2p - 2)$ spaces of $M(\mathbb{Z}/(p^n))$. Thus,

$$\begin{aligned} \pi_k^s(M_1) &= \pi_{k-2m}(M(\mathbb{Z}/(p^n))) \\ \pi_k^s(M_2) &= \pi_{k-2m-2p+2}(M(\mathbb{Z}/(p^n))). \end{aligned}$$

By Proposition 3.1.1, the first p-torsion in π_k^s is a copy of $\mathbb{Z}/(p)$ in degree $k = 2p - 3$. Therefore, the first two non-zero stable homotopy groups of M_1 and M_2 are

$$\begin{aligned} \pi^s_{2m}(M_1) &= \mathbb{Z}/(p^n) \\ \pi^s_{2m+2p-3}(M_1) &= \mathbb{Z}/(p) \\ \pi^s_{2m+2p-2}(M_2) &= \mathbb{Z}/(p^n) \\ \pi^s_{2m+4p-5}(M_2) &= \mathbb{Z}/(p). \end{aligned}$$

Using the exact sequence (3.1), it follows that the first non-zero stable homotopy group of X_m is

$$\pi^s_{2m}(X_m) = \mathbb{Z}/(p^n).$$

The next potentially non-zero stable homotopy group fits into the exact sequence (3.1) at degree $2m + 2p - 3$:

$$\mathbb{Z}/(p^n) \to \mathbb{Z}/(p) \to \pi^s_{2m+2p-3}(X_m) \to 0.$$

It follows that

$$\pi^s_k(\Sigma B\mathbb{Z}/(p^n)) = \begin{cases} \mathbb{Z}/(p^n) & \text{if } 0 < k < 2p-1 \text{ and } k \text{ is even,} \\ 0 & \text{if } 0 < k < 2p-1 \text{ and } k \text{ is odd.} \end{cases}$$

The theorem follows immediately. □

Corollary 3.1.3. *If,*

$$G = \mathbb{Z}/(n) = \bigoplus_{q|n} \mathbb{Z}/(q^{v_q(n)}),$$

then for $0 < k < 2\min_{q|n}(q) - 2$*,* $\pi^s_k(BG) \cong G$ *when* k *is odd and* $\pi^s_k(BG) = 0$ *when* k *is even.*

Proof. This follows from the propositions, since

$$BG \xrightarrow{\sim} \vee_{q|n} B\mathbb{Z}/(q^{v_q(n)}).$$

□

Corollary 3.1.4. *Denote by* m_k *the exponent of* π^s_k *for* $k \geq 1$*. If* $G = \mathbb{Z}/(n) = \bigoplus_{q|n} \mathbb{Z}/(q^{v_q(n)})$*, and if* $\beta \in \mathrm{H}^2(U_{\text{ét}}, \mu_n)$*, then, for*

$$1 < j < 2\min_{q|n}(q) - 1,$$

the cohomology group $\mathrm{H}^k(U_{\text{ét}}, \pi_j(\mathbf{T}^\beta))$ *is annihilated by* $n \cdot m_{j-1}$ *when* j *is even and by* m_{j-1} *when* j *is odd.*

Proof. The stalk of $\pi_j(\mathbf{T}^\beta)$ is the stalk of $\mathcal{T}_{j-1}$, which is isomorphic to

$$\pi^s_{j-1}(B\mu_n(k(\overline{x}))) \oplus \pi^s_{j-1}.$$

The corollary now follows from the computation of $\pi^s_{j-1}(B\mathbb{Z}/(n))$ of Proposition 3.1.2. □

3.2 Obstruction Theory

Let U be a geometrically connected and quasi-connected scheme. Recall that there is a model category structure on presheaves of simplicial sets on the étale site of U where $f : X \to Y$ is a cofibration of $X(V) \to Y(V)$ is an inclusion of simplicial sets for all $V \to U$, and where $f : X \to Y$ is a weak equivalence (called a *local weak equivalence*) if it induces an isomorphism of homotopy sheaves

$$\pi_t(X, x_0) \to \pi_t(Y, f(x_0))$$

for all $x_0 \in X(U)_0$. This is the Joyal model category structure. See [19], or Section 4.6.

For any pointed simplicial presheaf X, let $X \to \mathbb{H}X$ denote a fibrant replacement in the Joyal model category structure. There are coskeleta functors on simplicial presheaves:

$$(\operatorname{cosk}_n X)(U) = \operatorname{cosk}_n(X(U)).$$

By setting

$$X(n) = \mathbb{H} \operatorname{cosk}_n \mathbb{H}X,$$

I obtain a tower of fibrations of simplicial presheaves

$$\cdots \to X(n+1) \to X(n) \to X(n-1) \to \cdots$$

such that the U-sections

$$\cdots \to \Gamma(U, X(n+1)) \to \Gamma(U, X(n)) \to \Gamma(U, X(n-1)) \to \cdots$$

form a tower of fibrations of simplicial sets. The spectral sequence associated to this tower (see [5]) is called the Brown-Gersten spectral sequence for X:

$$\mathrm{E}_2^{s,t} \cong \begin{cases} \mathrm{H}^s(U, \pi_t Y) & \text{if } t - s \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The abutment is

$$\mathrm{E}_2^{s,t} \rightharpoonup \mathbb{H}^{t-s}\Gamma(U, X) = \pi_{t-s}\Gamma(U, \mathbb{H}X).$$

See Section 4.1 for the notation, which signifies a weak sort of convergence. The differentials d_k are of degree $(k, k-1)$. For details on the Brown-Gersten spectral sequence, see the original paper [6], or see [14].

Definition 3.2.1. Let X be a simplicial presheaf. I define two subgroups (pointed subsets if $t = 0$) of $\mathrm{H}^0(U, \pi_t X(t)) \xrightarrow{\simeq} \mathrm{H}^0(U, \pi_t X)$. First, define

$$\mathrm{H}^0_{\mathrm{red}}(U, \pi_t X) = \operatorname{im}(\pi_t \Gamma(U, X) \to \mathrm{H}^0(U, \pi_t X(t))).$$

Second, define
$$\mathrm{H}^0_{\mathrm{lift}}(U, \pi_t X) = \mathrm{im}(\pi_t G \to \mathrm{H}^0(U, \pi_t X(t))),$$
where G is the inverse limit of the U-sections of the Postnikov tower for X, and the map is induced by $G \to \Gamma(U, X(t))$ and sheafification:
$$\pi_t G \to \pi_t \Gamma(U, X(t)) \to \Gamma(U, \pi_t X(t)).$$

Theorem 3.2.2. *There are natural inclusions*
$$\mathrm{H}^0_{\mathrm{red}}(U, \pi_t X) \subseteq \mathrm{H}^0_{\mathrm{lift}}(U, \pi_t X)$$

Proof. The commutative diagram
$$\begin{array}{ccccc}
\pi_t\Gamma(U, X) & \longrightarrow & \pi_t\Gamma(U, \mathbb{H}X) & \longrightarrow & \pi_t G \\
& & & & \downarrow \\
& & & & \pi_t\Gamma(U, X(t)) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(U, \pi_t X) & \xrightarrow{\simeq} & \Gamma(U, \pi_t \mathbb{H}X) & \xrightarrow{\simeq} & \Gamma(U, \pi_t X(t))
\end{array}$$
shows that $\mathrm{H}^0_{\mathrm{red}}(U, \pi_t X) \subseteq \mathrm{H}^0_{\mathrm{lift}}(U, \pi_t X)$. □

Corollary 3.2.3. *A necessary condition for an element of $\mathrm{H}^0(U, \pi_t X)$ to lift to an element of $\pi_t\Gamma(U, X)$ is for it to be annihilated by all differentials.*

Remark 3.2.4. For $t = 0$, this condition is trivial, since $d_k = 0$ on $\mathrm{H}^0(U, \pi_0 X)$ for $k \geq 2$. For $t > 0$, $d_j : \mathrm{E}_j^{0,-t} \to \mathrm{E}_j^{0+j,-t-j+1}$, and $j - t - j + 1 \leq 0$ if and only if $-t + 1 \leq 0$.

Theorem 3.2.5 (Obstruction). *Let $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$, where U is a geometrically connected quasi-separated scheme. Fix a class $m \in \mathrm{H}^0(U, \mathbb{Z})$. A necessary condition for α to be represented by an Azumaya algebra of rank m^2 is that $d_k^\alpha(m) = 0$ for all $k \geq 2$, where the differentials d_k^α are those of the Brown-Gersten spectral sequence for $\mathbf{K}^\alpha$, the presheaf of twisted K-theory spaces. If, for some m with $n|m$, the differential $d_k(m)$ is non-torsion for some k, then α is not in the image of the Brauer group.*

Proof. Suppose that α is represented by an Azumaya algebra $\mathcal{A}$. Then, there exists an α-twisted locally free and finite rank sheaf $\mathcal{E}$ that is defined on all of U and such that $\mathcal{A} \cong \mathrm{End}(\mathcal{E})$. In particular, if $\mathcal{A}$ is of rank m^2, then $\mathcal{E}$ is of rank m. Therefore, there is a rank m element in $\pi_1^p \mathbf{K}^\alpha(U)$. This maps to m in $\mathrm{H}^0(U_{\text{ét}}, \pi_1 \mathbf{K}^\alpha)$, which is isomorphic to $\mathrm{H}^0(U_{\text{ét}}, \mathbb{Z})$. Therefore, by Theorem 3.2.2, m lies in $\mathrm{H}^0_{\text{red}}(U_{\text{ét}}, \pi_1 \mathbf{K}^\alpha)$, and hence in $\mathrm{H}^0_{\text{lift}}(U_{\text{ét}}, \pi_1 \mathbf{K}^\alpha)$. It follows that

$$d_k^\alpha(m) = 0$$

for $k \geq 2$ in the Brown-Gersten spectral sequence for $\mathbf{K}^\alpha$. This completes the proof. □

3.3 The Period and the Spectral Index

Definition 3.3.1. Let $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$, where U is of finite étale cohomological dimension. Then, there is a unique smallest positive integer $spi(\alpha)$ such that

$$d_k^\alpha(spi(\alpha)) = 0$$

for all $k \geq 2$. Call this the **spectral index**. By the **obstruction** property, if it is finite, then it is the smallest integer that *might* be the rank of an α-twisted locally free finite rank sheaf. By Theorem 3.2.5,

$$spi(\alpha) | ind(\alpha),$$

and by the results of Chapter 4,

$$per(\alpha) | spi(\alpha).$$

Example 3.3.2. If D is a cyclic division algebra $(x, y)_{\zeta_n}$, so that $per(D) = ind(D) = n$, then the aforementioned results imply that $spi(D) = n$.

Example 3.3.3. If D/k is a division algebra, and if l/k is a finite separable field extension of degree prime to $per(D)$, then a standard argument using norm maps says that $spi(D_l) = spi(D)$.

Denote by m_j the exponent of π_j^s, the jth stable homotopy group of S^0, and let n_j^α denote the exponent of $\pi_j^s(B\mathbb{Z}/(per(\alpha)))$. Finally, let l_j^α denote the exponent of $\pi_j^s \oplus \pi_j^s(B\mathbb{Z}/(per(\alpha)))$. So, l_j^α is the least common multiple of m_j and n_j^α.

Theorem 3.3.4 (Bound). *Let U be a geometrically connected quasi-separated scheme such that the étale cohomological dimension of U with coefficients in finite sheaves is a finite integer d. Let $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)_{\text{tors}}$. Then,*

$$spi(\alpha) | \prod_{j \in \{1, \ldots, d-1\}} l_j^\alpha.$$

Proof. Let β be a lift of α to $\mathrm{H}^2(U_{\text{ét}}, \mu_{per(\alpha)})$, and let d_k^β denote the kth differential in the Brown-Gersten spectral sequence for $\mathbf{T}^\beta$. As the class 1 in $\mathrm{H}^0(U_{\text{ét}}, \pi_1(\mathbf{T}^\beta))$ maps to the class 1 in $\mathrm{H}^0(U_{\text{ét}}, \pi_1(\mathbf{K}^\alpha))$, if $d_k^\beta(m) = 0$, then $d_k^\alpha(m) = 0$. The differential d_k^β lands in a subquotient of $\mathrm{H}^k(U, \pi_k(\mathbf{T}^\beta))$. Therefore, since $\pi_k(\mathbf{T}) = \mathcal{T}_{k-1}$, d_k^β lands in a group of exponent at most l_{k-1}^α, by Corollary 3.1.4. As the differentials d_k^β all vanish for $k > d$, the theorem follows. □

Corollary 3.3.5. *Let U be a geometrically connected quasi-separated scheme of finite l-torsion étale cohomological dimension. Then, $spi(\alpha)$ is finite for all α with l-power period.*

Example 3.3.6. Let Q be the non-separated quadric with α the non-zero cohomological Brauer class [13]. Then $per(\alpha) = spi(\alpha) = 2$, while $ind(\alpha) = +\infty$. Note that Q *is* quasi-separated.

Definition 3.3.7. Let k be a field, and let S be a non-empty set of primes. Let $cd_S k$ be the supremum of all the cohomological dimensions $cd_q k$ for all primes $q \in S$.

Theorem 3.3.8. *Let k be a field, and let $\alpha \in \mathrm{H}^2(k, \mathbb{G}_m)$. Let S be the set of prime divisors of $per(\alpha)$, and suppose that $d = cd_S k$ satisfies $d = 2c$ or $d = 2c+1$ and that $d < 2\min_{q\in S}(q)$. Then,*

$$spi(\alpha)|(per(\alpha))^c.$$

Proof. Combining Theorem 3.3.4 and Corollary 3.1.4, it follows that, if d is even, then

$$d_k^\beta(an^c) = 0$$

for all $k \geq 2$, where a is prime to n. The same reasoning shows that if d is odd, then

$$d_k^\beta(an^c) = 0$$

when $2 \leq k \leq d-1$. By [26], the stalks of $\mathcal{K}_{2j}^\alpha$ are torsion-free for $j > 0$. Therefore, the maps

$$\mathrm{H}^2(k, \mathcal{T}_{2j}) \to \mathrm{H}^2(k, \mathcal{K}_{2j})$$

are zero for $j > 0$. It follows that if $d_k^\beta(m) = 0$ for $2 \leq k \leq 2j$, then $d_k^\alpha(m) = 0$ for $2 \leq k \leq 2j+1$. Therefore, when d is odd,

$$d_k^\beta(an^c) = 0$$

for $2 \leq k \leq d$ and hence for all $k \geq 2$.

Thus,

$$spi(\alpha)|an^c,$$

where a is relatively prime to n. On the other hand, as k is a field, the primes divisors of $per(\alpha)$ and $spi(\alpha)$ are the same. So,

$$spi(\alpha)|n^f$$

for some positive integer f. Now, as $\mathrm{H}^0_{\mathrm{lift}}(U_{\text{ét}}, \pi_1(\mathbf{K}^\alpha))$ is cyclic, it follows that

$$spi(\alpha)|n^{\min(c,f)}|n^c.$$

This completes the proof.

□

The condition $d < 2\min_S(q)$ excludes no primes for function fields of curves, surfaces, or three-folds. It excludes the prime 2 for function fields of four-folds and five-folds.

Let

$$\mathbf{K}_0^\alpha(X)^{(0)} = \mathbf{K}_0^\alpha / \ker\left(\mathbf{K}_0^\alpha(X) \xrightarrow{rank} \mathbb{Z}\right)$$
$$\mathbf{K}_0^{\alpha,\text{ét}}(X)^{(0)} = \mathbf{K}_0^{\alpha,\text{ét}} / \ker\left(\mathbf{K}_0^{\alpha,\text{ét}}(X) \xrightarrow{rank} \mathbb{Z}\right).$$

When α is trivial, the natural inclusion

$$\mathbf{K}_0^\alpha(X)^{(0)} \to \mathbf{K}_0^{\alpha,\text{ét}}(X)^{(0)} \tag{3.2}$$

is an isomorphism.

Corollary 3.3.9. *The map of Equation (3.2) is not surjective in general when α is not trivial.*

Proof. For example, let $k(C)$ be the function field of a curve over a p-adic field. Jacob and Tignol have shown in an appendix of [25] that there are division algebras over $k(C)$ for which $ind(\alpha) = per(\alpha)^2$. However, since these fields are of cohomological dimension 3, it follows that $spi(\alpha) = per(\alpha)$. Thus, the map is not surjective for $X = \operatorname{Spec} k(C)$. □

Chapter 4

Divisibility

The fundamental technical tool, developed in this chapter, used to show that the period divides the spectral index, is a Čech approximation to the Brown-Gersten spectral sequence. Let U be a geometrically connected and quasi-separated scheme, let $\mathcal{U}^\bullet \to U$ be an étale hypercover, and let $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$. Then, there is a morphism of spectral sequences

$$^{\mathcal{U}^\bullet}\,\mathrm{E}_2\,\mathbf{K}^\alpha \to {}^{\mathbf{BG}}\,\mathrm{E}_2\,\mathbf{K}^\alpha$$

which on E_2 terms is the natural map

$$\check{\mathrm{H}}^s(\mathcal{U}^\bullet, \mathbf{K}^\alpha_j) \to \mathrm{H}^s(U_{\text{ét}}, \mathcal{K}^\alpha_t),$$

where $\mathbf{K}^\alpha_t$ is the presheaf of twisted K-groups. See Theorem 4.8.1. In Sections 4.11 and 4.12, the relevant differentials of the Čech approximation are computed. This allows a computation, in Theorem 4.13.1, of $d_2^\alpha(1)$ in the descent spectral sequence for twisted K-theory. This establishes the **divisibility** theorem.

The Chapter begins with preliminaries, on spectral sequences associated to fibrations, in Section 4.1, and on cosimplicial spaces and their spectral sequences, in Sections 4.2, 4.3, and 4.4. These spectral sequences are compared in Section 4.5. There, the Postnikov and homotopy limit spectral sequences for a cosimplicial space are shown to be isomorphic after E_2. This theorem seems to be well-known, but I know of no reference for a proof.

What follows are preliminary sections on presheaves of simplicial spaces, in Section 4.6, and their spectral sequences, in Section 4.7. Then, the Čech approximation morphism is established. Sections 4.10, 4.11, and 4.12 are dedicated to computing some differentials. Finally, in Section 4.13, the computation of $d_2^\alpha(1)$ is made, and the **divisibility** property established:

$$per(\alpha)|spi(\alpha).$$

The computation of $d_2^\alpha(1)$ is also in a preprint of Kahn and Levine [20, Proposition 6.9.1]. Their proof is entirely different, using motivic slices.

4.1 Spectral Sequences Associated to Towers of Fibrations

Construction 4.1.1. Let

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to *$$

be a tower of fibrations of pointed spaces. Let F_s be the fiber of $X_s \to X_{s-1}$, and denote by X the limit of the tower.

There are long exact sequences associated to the fibrations $F_s \to X_s \to X_{s-1}$:

$$\cdots \to \pi_{t-s}F_s \xrightarrow{k} \pi_{t-s}X_s \xrightarrow{i} \pi_{t-s}X_{s-1} \xrightarrow{j} \pi_{t-s-1}F_s \to \cdots . \tag{4.1}$$

These sequences continue all the way down to $\pi_0 X_{s-1}$:

$$\pi_2 X_{s-1} \xrightarrow{j} \pi_1 F_s \xrightarrow{k} \pi_1 X_s \xrightarrow{i} \pi_1 X_{s-1} \xrightarrow{j} \pi_0 F_s \xrightarrow{k} \pi_0 X_s \xrightarrow{i} \pi_0 X_{s-1},$$

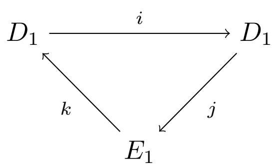
and

$$\pi_1 X_{s-1} \xrightarrow{j} \pi_0 F_s$$

extends to an action of $\pi_1 X_{s-1}$ on $\pi_0 F_s$ so that j is the map onto the orbit of the basepoint of F_s under the action of $\pi_1 X_{s-1}$. Besides the usual conditions of $\ker = \mathrm{im}$ in the range that this makes sense, the exactness of Equation (4.1) *means*

- that the quotient of $\pi_0 F_s$ under this action injects into $\pi_0 X_s$,
- that the cokernel (quotient set) of $\pi_0 F_s \xrightarrow{k} \pi_0 X_s$ injects into $\pi_0 X_{s-1}$,
- that the stabilizer of the action of $\pi_1 X_{s-1}$ on $\pi_0 F_s$ at the base-point of F_s is the quotient of $\pi_1 X_s$ by the image of $\pi_1 F_s \to \pi_1 X_s$, and
- that $\pi_2 X_{s-1}$ maps to the center of $\pi_1 F_s$.

The tower of fibrations and the exact sequences above define an exact couple



where D and E are bigraded groups:

$$\begin{aligned}\mathrm{D}_1^{s,t} &= \pi_{t-s}X_s \\ \mathrm{E}_1^{s,t} &= \pi_{t-s}F_s.\end{aligned}$$

The maps i, j, and k are of bi-degrees $(-1,1)$, $(1,-2)$, and $(0,0)$:

$$\begin{aligned}i &: \pi_{t-s}X_s \to \pi_{t-s}X_{s-1} \\ j &: \pi_{t-s}X_{s-1} \to \pi_{t-s-1}F_s \\ k &: \pi_{t-s-1}F_s \to \pi_{t-s-1}X_s.\end{aligned}$$

As usual, the exact couple gives rise to a differential $d = j \circ k$ on E. It is of bi-degree $(1,-2)$. The first derived exact couple is

$$\begin{aligned}\pi_{t-s}X_s^{(1)} &= \mathrm{D}_2^{st} = \mathrm{im}(i) = \mathrm{im}(\pi_{t-s}X_{s+1} \xrightarrow{i} \pi_{t-s}X_s) \subseteq \pi_{t-s}X_s \\ \pi_{t-s}F_s^{(1)} &= \mathrm{E}_2^{st} = \mathrm{H}(d) = \ker(\pi_{t-s}F_s \xrightarrow{k} \pi_{t-s}X_s/\,\mathrm{im}(i))/ \\ &\qquad \ker(\pi_{t-s+1}X_{s-1} \xrightarrow{i} \pi_{t-s+1}X_{s-2}).\end{aligned}$$

When $s = t$, then the definition of E_2^{st} should be interpreted as the quotient of the pointed set

$$\ker(\pi_0 F_s \xrightarrow{k} \pi_0 X_s/\,\mathrm{im}(i))$$

by the action of $\ker(\pi_1 X_{s-1} \xrightarrow{i} \pi_1 X_{s-2}) \subseteq \pi_1 X_{s-1}$. Then, the sequences

$$\pi_2 X_{s-1}^{(1)} \xrightarrow{j} \pi_1 F_s^{(1)} \xrightarrow{k} \pi_1 X_s^{(1)} \xrightarrow{i} \pi_1 X_{s-1}^{(1)} \xrightarrow{j} \pi_0 F_s^{(1)} \xrightarrow{k} \pi_0 X_s^{(1)} \xrightarrow{i} \pi_0 X_{s-1}^{(1)},$$

are also exact in the generalized sense above.

Repeating this process, one obtains a generalized spectral sequence $\mathrm{E}_1^{s,t}\{X_*\}$ associated to the tower, with

$$\mathrm{E}_1^{s,t} = \pi_{t-s}F_s \rightharpoonup \pi_{t-s}X,$$

where F_s is the fiber of $X_n \to X_{n-1}$. The differential d_r is of degree $(r, r-1)$. See [5, Chapter IX]. The harpoon $\rightharpoonup$ means that the spectral sequence may not converge in the usual sense. Instead, there is a filtration

$$Q_s\pi_i X = \ker(\pi_i X \to \pi_i X_s)$$

with successive quotients

$$e_\infty^{s,t} = \ker(Q_s\pi_{t-s}X \to Q_{s-1}\pi_{t-s}X),$$

and inclusions

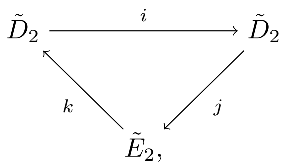
$$e_\infty^{s,t} \subseteq \mathrm{E}_\infty^{s,t}\,. \tag{4.2}$$

Write $\mathrm{E}_1^{s,t} \Rightarrow \pi_{t-s}X$ when the spectral sequence does hold in the usual sense, in which case equality holds in Equation (4.3). In this case, say that the spectral sequence *converges completely*.

There is a second spectral sequence $\tilde{\mathrm{E}}_2^{s,t}\{X_*\}$, which is simply a re-indexed version of the first:

$$\tilde{\mathrm{E}}_2^{s,t} = \mathrm{E}_1^{t,2t-s} = \pi_{t-s}F_t \rightharpoonup \pi_{t-s}X.$$

This is derived from the exact couple



With $\tilde{\mathrm{E}}_2$ as above, and

$$\tilde{\mathrm{D}}_2^{s,t} = \pi_{t-s}X_t.$$

The filtration is the same:

$$\tilde{Q}_s\pi_i X = Q_s\pi_i X,$$

but the successive quotients are

$$\tilde{e}_\infty^{s,t} = e_\infty^{t,2t-s}.$$

Here, the differentials are also of degree $(r, r-1)$.

The spectral sequence and the filtration $Q_s\pi_i$ are functorial for towers of pointed spaces. We view the spectral sequence as including the information of the abutment and the filtration on the abutment. Thus, a morphism of spectral sequences includes a filtration-respecting morphism of the abutment.

Remark 4.1.2. These spectral sequences are generalized in the sense that not every term is an abelian group. Indeed, $\mathrm{E}_1^{s,t}$ is a group when $t-s=1$ and it is a pointed set when $t-s=0$. Nonetheless, there is a good notion of such spectral sequences. For details, see [5, Section IX.4]. Under certain conditions, these spectral sequence do converge in some range to some of the homotopy groups of X. For instance, suppose that $i \geq 1$ and that for each $s \geq 0$ there is an integer $N(s) \geq 1$ such that

$$\mathrm{E}_M^{s,s+j} = \mathrm{E}_\infty^{s,s+j} \tag{4.3}$$

for all $M \geq N(s)$ when $j = i$ and $j = i + 1$. Then, $\mathrm{E}_*^{s,t}\{X_*\}$ converges to $\pi_i X$. This is also true for $i = 0$ when all of the homotopy sets of the spaces in the tower are abelian groups. Again, for details see [5, Section IX.5].

Note that in the application to K-theory, the spaces in the tower will have homotopy sets π_t which are abelian groups for all $t \geq 0$. Moreover, the convergence conditions of Equation (4.3) under the finite cohomological dimension conditions used in this paper.

4.2 Cosimplicial Spaces

Definition 4.2.1. Let Δ be the category of finite simplices. Objects of Δ are non-empty finite ordered sets, and morphisms are set morphisms that preserve order. The category $\mathbf{sSets}$ of simplicial sets is the functor category $\mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{Sets})$. In general, if C is a category, then sC is the category $\mathbf{Fun}(\Delta^{\mathrm{op}}, C)$, the category of simplicial objects in C. Objects of $\mathbf{sSets}$ will be called spaces. The category $\mathbf{sSets}_*$ is the category of pointed spaces.

Definition 4.2.2. If C is a category, then denote by $\mathbf{c}C$ the functor category $\mathbf{Fun}(\Delta, C)$, the category of cosimplicial objects in C. The category of cosimplicial spaces is the category $\mathbf{csSets}$. Write $\mathbf{csSets}_*$ for the category of cosimplicial pointed spaces.

Example 4.2.3. For a space X, let the same symbol X denote the constant cosimplicial space $n \mapsto X$.

Example 4.2.4. The cosimplicial space Δ is the functor $n \mapsto \Delta^n$, where Δ^n is the simplicial space Δ^n.

Example 4.2.5. Let $\mathcal{U}^\bullet$ be a hypercover in a site C, and let X be a presheaf of simplicial sets on C (a presheaf of spaces). Then, $X_{\mathcal{U}^\bullet}$, denotes the cosimplicial space given by evaluating X at each level of $\mathcal{U}^\bullet$.

Definition 4.2.6. If F is an endofunctor of $\mathbf{sSets}$, then one extends F to an endofunctor on $\mathbf{csSets}$ by level-wise application. That is, for a cosimplical space X, define $F(X)^n = F(X^n)$. The typical examples are the s-skeleton functors $X \mapsto X[s]$ and the Ex-functor.

Definition 4.2.7. Let $\mathbf{P}$ be a property of spaces. Then, a cosimplicial space X is level $\mathbf{P}$ if each space X_n is $\mathbf{P}$ for $n \geq 0$. Similarly, if $\mathbf{Q}$ is a property of morphisms of spaces, then a morphism $f : X \to Y$ of cosimplicial spaces is level $\mathbf{Q}$ if $f^n : X^n \to Y^n$ is $\mathbf{Q}$ for all $n \geq 0$.

There is a good model structure, the Reedy structure, on cosimplicial spaces. Let X be a cosimplicial space so that X^n is a simplicial set for $n \geq 0$. A morphism $f : X \to Y$ is a weak equivalence if each $f^n : X^n \to Y^n$ is a weak equivalence; that is, if f is a level weak equivalence. The maximal augmentation of a cosimplicial space is the simplicial set that equalizes $d^0, d^1 : X^0 \to X^1$. A map f of cosimplicial spaces is called a cofibration if it is a level cofibration (level monomorphism) and if it induces an isomorphism on the maximal augmentations. The fibrations are all those morphisms with the right lifting property with respect to acyclic cofibrations. A proof that this is a model category may be found in [5, section X.5].

Example 4.2.8. As examples of cofibrant objects, consider Δ and $\Delta[s]$. Indeed, Δ^0 is a single point, and Δ^1 is the 1-simplex. The coface maps d^0 and d^1 send the unique point of Δ^0 to the vertices 1 and 0 respectively of Δ^1. Therefore, the maximal augmentation is the empty simplicial complex. This also shows that $\Delta[s] \to \Delta$ is a cofibration.

Let X be a cosimplicial space. Then, define the nth matching object of X to be

$$M^n X = \varprojlim_{\phi:\mathbf{n}\to\mathbf{k}} X^k,$$

where ϕ runs over all surjections $\mathbf{n} \to \mathbf{k}$ in Δ. There is a natural map $X^{n+1} \to M^n X$.

Proposition 4.2.9 ([5, Section X.4.5])**.** *A morphism $f : X \to Y$ is a fibration if and only if the induced map*

$$X^{n+1} \to Y^{n+1} \times_{M^n Y} M^n X$$

is a fibration of simplicial sets for all $n \geq -1$.

The closed model structure on cosimplicial spaces is simplicial. That is, there is a functor

$$\mathbf{Map} : \mathbf{csSets}^{\mathrm{op}} \times \mathbf{csSets} \to \mathbf{sSets}$$

defined by

$$\mathbf{Map}(X, Y) : n \mapsto \mathrm{Hom}(X \times \Delta^n, Y).$$

The space $\mathbf{Map}(X, Y)$ is called the function complex from X to Y. Similarly, if X and Y are cosimplicial pointed spaces, then there is a pointed function complex

$$\mathbf{Map}_*(X, Y) \in \mathbf{sSets}_*$$

defined by

$$\mathbf{Map}_*(X, Y)_n = \mathrm{Hom}(X \wedge \Delta^n_+, Y),$$

where S^n is the simplicial complex $\Delta^n/\partial\Delta^n$.

Proposition 4.2.10 ([5, Section X.5])**.** *The simplicial model category axiom SM7 is satisfied. Namely, if $A \to B$ is a cofibration of cosimplicial spaces and if $X \to Y$ is a fibration of cosimplicial spaces, then*

$$\mathbf{Map}(B, X) \to \mathbf{Map}(A, X) \times_{\mathbf{Map}(A,Y)} \mathbf{Map}(B, Y)$$

is a fibration.

There is a functor $\mathbf{csSets} \to \mathbf{csSets}_*$ defined by

$$X = (n \mapsto X^n) \mapsto X_+ = (n \mapsto X^{n,+}),$$

where $X_{n,+}$ is the space X_n with a disjoint basepoint.

Definition 4.2.11. For each integer $n \geq 2$, there is a functor

$$\pi_n : \mathbf{csSets}_* \to \mathbf{cAb},$$

where $\mathbf{cAb}$ is the category of cosimplicial abelian groups, defined by

$$\pi_n(X)^m = \pi_n(X^m).$$

There are also functors, defined by the same equation,

$$\begin{aligned} \pi_1 &: \mathbf{csSets}_* \to \mathbf{cGroups} \\ \pi_0 &: \mathbf{csSets}_* \to \mathbf{cSets}_*, \end{aligned}$$

where $\mathbf{cSets}_*$ is the category of cosimplicial pointed sets and $\mathbf{cGroups}$ is the category of cosimplicial groups.

Definition 4.2.12. Let A be a cosimplicial abelian group, cosimplicial group, or cosimplicial pointed set. A pointed cosimplicial space X is called a $K(A, n)$-cosimplicial space if $\pi_n X \cong A$, while $\pi_m X \cong *$ for $m \neq n$.

4.3 Two Spectral Sequences for Cosimplicial Spaces

Let X be a pointed cosimplicial space. Define pointed simplicial complexes

$$\begin{aligned} \mathbf{Tot}_\infty X &= \mathbf{Map}_*(\Delta_+, X), \\ \mathbf{Tot}_s X &= \mathbf{Map}_*(\Delta[s]_+, X). \end{aligned}$$

By axiom SM7 and Example 4.2.8, if X is fibrant, then $\mathbf{Tot}_s X \to \mathbf{Tot}_{s-1} X$ gives a tower of pointed fibrations. The inverse limit of this tower is $\mathbf{Tot}_\infty X$ when X is fibrant.

Definition 4.3.1. For an arbitrary cosimplicial pointed space X let $X \to \mathbb{H}_c X$ be a pointed fibrant resolution. Then, the total space spectral sequence of X, ${}^{\mathbf{T}}\mathrm{E}_1\, X$, is defined to be the spectral sequence of the tower $\mathbf{Tot}_* \mathbb{H}_c X$:

$${}^{\mathbf{T}}\mathrm{E}_1^{s,t}\, X = \mathrm{E}_1^{s,t}\{\mathbf{Tot}_* \mathbb{H}_c X\} \rightharpoonup \pi_{t-s} \mathbf{Tot}_\infty \mathbb{H}_c X.$$

The fiber F_s of $\mathbf{Tot}_s \mathbb{H}_c X \to \mathbf{Tot}_{s-1} \mathbb{H}_c X$ and the homotopy groups of the fiber F_s are identified in [5, Proposition X.6.3] (see also Section 4.9):

$$F_s \simeq \mathbf{Map}_*(S^s, NX^s),$$

where NX^s is the fiber of the fibration $\mathbb{H}_c X^s \to M^{s-1}\mathbb{H}_c X$; see Proposition 4.2.9. Moreover,

$$\pi_i NX^s \cong \pi_i \mathbb{H}_c X^s \cap \ker s^0 \cap \cdots \cap \ker s^{s-1},$$

where the maps s^i are the cosimplicial degeneracies:

$$s^i : \mathbb{H}_c X^s \to \mathbb{H}_c X^{s-1}.$$

Therefore, there is a natural identification

$$^{\mathbf{T}}\,\mathrm{E}_1^{s,t}\, X \cong \pi_t \mathbb{H}_c X^s \cap \ker s^0 \cap \cdots \cap \ker s^{s-1} \qquad t \geq s \geq 0.$$

Note that for $t \geq 2$, $^{\mathbf{T}}\,\mathrm{E}_1^{s,t}\, X$ is the s-degree of the normalized cochain complex $N^*\pi_t \mathbb{H}_c X$ associated to $\pi_t \mathbb{H}_c X$. It is tedious but not hard to check that under this identification, the differential d_1 of the spectral sequence is chain homotopic to the differential of the normalized cochain complex. Therefore, there are natural identifications

$$^{\mathbf{T}}\,\mathrm{E}_2^{s,t}\, X \cong \mathrm{H}^s(N^*\pi_t \mathbb{H}_c) \cong \mathrm{H}^s(C^*\pi_t \mathbb{H}_c X) \cong \mathrm{H}^s(C^*\pi_t X) \qquad t \geq 2,\ t \geq s \geq 0,$$

where $C^*\pi_t X$ denotes the unnormalized cochain complex associated to the cosimplicial abelian group $\pi_t X$.

It is not hard to extend these identifications for $t = 0$ and $s = 0$, and for $t = 1$ and $s = 0, 1$. For a detailed discussion, see [5, Section X.7].

Define, for a cosimplicial abelian group A, the sth cohomotopy group for $s \geq 0$ as

$$\pi^s A = \mathrm{H}^s(C^* A).$$

If G is a cosimplicial pointed set or cosimplicial group, then define $\pi^0 G$ as the equalizer of $\partial^0, \partial^1 : G^0 \rightrightarrows G^1$. This is a group if G is.

Similarly, define a pointed cohomotopy set $\pi^1 G$ for G a cosimplicial group as follows. Let

$$Z^1 G = \{g \in G^1 : (\partial^0 g)(\partial^1 g)^{-1}(\partial^2 g) = 1\}.$$

There is an action of $G^0 \times Z^1 G \to Z^1 G$ given by

$$(g_0, g_1) \mapsto (\partial^1 g_0) g_1 (\partial^0 g_0)^{-1}.$$

The set Z^1G is pointed by the element $1 \in G^1$. Let $\pi^1 G$ be the quotient set of Z^1G by this action, pointed by the orbit of $1 \in Z^1G$. Then, by [5, Paragraph X.7.2], there are natural identifications

$$^{\mathbf{T}}\,\mathrm{E}_2^{s,t}\,X \cong \begin{cases} \pi^s\pi_t X & \text{if } t \geq s \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

There is another spectral sequence for cosimplicial spaces, which is useful when X is level Kan. This is the homotopy limit spectral sequence, which, in fact, exists in much greater generality. See [5, Chapter XI]. The main tool is a functor from cosimplicial spaces to cosimplicial spaces called Π. Let $N\Delta$ be the nerve of the category Δ. Then an element of $N\Delta_n$ is a simplex $\mathbf{i_0} \to \cdots \to \mathbf{i_n}$. For X an arbitrary cosimplicial space, let ΠX denote the space whose nth level is

$$\Pi^n X = \prod_{\mathbf{i}_* \in N\Delta_n} X^{i_n}.$$

The face map ∂^j for $j < n$ composed with projection onto $\mathbf{i}_*$ is projection onto $\partial_j(\mathbf{i}_*)$ followed by the identity. The face map ∂^n composed with projection onto $\mathbf{i}_*$ is projection onto $\partial_n(\mathbf{i}_*)$ followed by $X(\mathbf{i_{n-1}} \to \mathbf{i_n})$. Similarly, the degeneracy s^j followed by projection onto $\mathbf{i}_*$ is projection onto $s_j(\mathbf{i}_*)$ followed by the identity.

The important thing about the cosimplicial replacement functor Π is that it takes level fibrations into cosimplicial fibrations and preserves weak equivalences. See [5, Proposition X.5.3].

Definition 4.3.2. Let X be a cosimplicial pointed space. Let $\mathrm{Ex}^\infty X$ denote the cosimplicial pointed space obtained from X by applying the Ex^∞-functor to each level. Then, $\Pi\,\mathrm{Ex}^\infty X$ is fibrant. Define the homotopy limit spectral sequence of X, $^{\mathbf{HL}}\,\mathrm{E}_1\,X$, to be

$$^{\mathbf{HL}}\,\mathrm{E}_1^{s,t}\,X = \mathrm{E}_1^{s,t}\{\mathbf{Tot}_*\Pi\,\mathrm{Ex}^\infty X\} \rightharpoonup \pi_{t-s}\mathbf{Tot}_\infty\Pi\,\mathrm{Ex}^\infty X.$$

The space $\mathbf{Tot}_\infty\Pi\,\mathrm{Ex}^\infty X$ is called the *homotopy limit* of X, and will be written as $\mathrm{holim}_\Delta X$.

Lemma 4.3.3. *If X is a cosimplicial pointed space that is level Kan, then the natural morphism $X \to \mathrm{Ex}^\infty X$ induces an isomorphism of spectral sequences*

$$\mathrm{E}_1^{s,t}\{\mathbf{Tot}_*\Pi X\} \xrightarrow{\cong} \mathrm{E}_1^{s,t}\{\mathbf{Tot}_*\Pi\,\mathrm{Ex}^\infty X\}.$$

This morphism is natural in morphisms of cosimplicial pointed level Kan spaces.

Let X be an arbitrary pointed cosimplicial space. The functor Π can be defined on cosimplicial objects in any category with finite products. In particular, on pointed sets, groups, and abelian groups. There are natural isomorphisms, of cosimplicial pointed sets for $n = 0$, cosimplicial groups for $n = 1$, and cosimplicial abelian groups for $n > 1$,

$$\pi_n \Pi X \simeq \Pi \pi_n X,$$

where $\pi_n X$ is the cosimplicial object obtained by evaluating π_n at each cosimplicial level.

For X a cosimplicial object in a category with finite products, there is an natural morphism $X \to \Pi X$. The maps

$$X^n \to \prod_{\mathbf{i}_* \in N\Delta_n} X^{i_n}$$

are described as follows. The simplex $\mathbf{i}_*$ determines a morphism $\mathbf{n} \to \mathbf{i_n}$, by taking the images of 0 from each i_i, $0 \leq i < n$. This induces the map X^n to the product, and it extends to a cosimplicial map $X \to \Pi X$.

Proposition 4.3.4 ([5, Paragraph XI.7.3]). *The canonical map $C^* \pi_n X \to C^* \Pi \pi_n X$ is a quasi-isomorphism.*

Proposition 4.3.5 ([5, Paragraph XI.7.5]). *If X is a fibrant cosimplicial pointed space, then the natural morphism*

$${}^{\mathbf{T}}\,\mathrm{E}_1\, X \to {}^{\mathbf{HL}}\,\mathrm{E}_1\, X$$

of spectral sequences is an isomorphism.

4.4 The Postnikov Spectral Sequence

Definition 4.4.1. Let X be a level-fibrant pointed cosimplicial space. Denote by $X(n)$ the level-wise application of the coskeleton functor. Then each $X \to X(n)$ and $X(m) \to X(n)$, $m \geq n$ is a level fibration. Therefore, $\Pi X(m) \to \Pi X(n)$ is a fibration for $m \geq n$. The spectral sequence of this tower is called the *Postnikov spectral sequence* for X:

$${}^{\mathbf{P}}\,\mathrm{E}_2^{s,t}\, X = \tilde{\mathrm{E}}_2^{s,t}\{\mathbf{Tot}_\infty \Pi X(*)\} \cong \pi_{t-s}\mathbf{Tot}_\infty G(t) \rightharpoonup \pi_{t-s}\,\mathrm{holim}_\Delta\, X,$$

where $G(t)$ is the fiber of $\Pi X(t) \to \Pi X(t-1)$. This fiber is a fibrant resolution of a cosimplicial $K(\pi_t X, t)$-space.

By [5, Paragraphs XI.7.2-3], there are natural isomorphisms

$$\pi_{t-s}\mathbf{Tot}_\infty G(t) \simeq \pi^s \pi_t X$$

for $t \geq s \geq 0$. Thus,

$${}^{\mathbf{HL}}\,\mathrm{E}_2^{s,t}\, X \cong {}^{\mathbf{P}}\,\mathrm{E}_2^{s,t}\, X.$$

In the next section, I will show that this extends to an isomorphism of spectral sequences.

4.5 Comparison Theorem

Theorem 4.5.1. *Let X be level Kan. Then, there is a natural isomorphism ϕ of spectral sequences*

$$\phi :^{\mathbf{HL}} \mathrm{E}_2\, X \to^{\mathbf{P}} \mathrm{E}_2\, X.$$

The left-hand side is the homotopy limit spectral sequence beginning with the E_2-page, while the right-hand side is the Postnikov tower spectral sequence.

Proof. Recall that to create an isomorphism of spectral sequences that come from exact couples

$$\begin{array}{ccc} {}^{I}\,\mathrm{D}_2 & \xrightarrow{\ i\ } & {}^{I}\,\mathrm{D}_2 \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & {}^{I}\,\mathrm{E}_2, & \end{array} \qquad \begin{array}{ccc} {}^{II}\,\mathrm{D}_2 & \xrightarrow{\ i\ } & {}^{II}\,\mathrm{D}_2 \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & {}^{II}\,\mathrm{E}_2 & \end{array}$$

it suffices to create a morphism of exact couples that is an isomorphism just on the E-terms:

$$\phi :^{I} \mathrm{E}_2 \xrightarrow{\simeq} {}^{II}\,\mathrm{E}_2.$$

Indeed, this is enough to guarantee that the morphism induces an isomorphism on $\mathrm{H(E)}$ and a morphism of the derived couples. So, it follows by induction that this is sufficient.

Since X is level Kan, there is a double tower of fibrations, of which a typical square is

$$\begin{array}{ccc} \mathbf{Tot}_{s+1}\Pi X(t-1) & \longleftarrow & \mathbf{Tot}_{s+1}\Pi X(t) \\ \downarrow & & \downarrow \\ \mathbf{Tot}_{s}\Pi X(t-1) & \longleftarrow & \mathbf{Tot}_{s}\Pi X(t). \end{array}$$

These fit into a bigger diagram

$$\begin{array}{ccccc} \mathbf{Tot}_{\infty}\Pi X(t-1) & \longleftarrow & \mathbf{Tot}_{\infty}\Pi X(t) & \longleftarrow & \mathbf{Tot}_{\infty}\Pi X \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Tot}_{s+1}\Pi X(t-1) & \longleftarrow & \mathbf{Tot}_{s+1}\Pi X(t) & \longleftarrow & \mathbf{Tot}_{s+1}\Pi X \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Tot}_{s}\Pi X(t-1) & \longleftarrow & \mathbf{Tot}_{s}\Pi X(t) & \longleftarrow & \mathbf{Tot}_{s}\Pi X. \end{array} \tag{4.5}$$

The horizontal inverse limits are $\mathbf{Tot}_s\Pi X$ and the vertical inverse limits are $\mathbf{Tot}_\infty\Pi X(t)$. Thus the homotopy limit spectral sequence comes the tower of fibrations at the horizontal limit, while the Postnikov spectral sequence comes from the tower of fibrations at the vertical limit. Define $F(s+1)$ to be the fiber of $\mathbf{Tot}_{s+1}\Pi X \to \mathbf{Tot}_s\Pi X$, and define $G(t)$ be the fiber of $\mathbf{Tot}_\infty\Pi X(t) \to \mathbf{Tot}_\infty\Pi X(t-1)$.

First, construct a morphism

$$^{\mathbf{HL}}\,\mathrm{D}_2^{s,t} = \mathrm{im}(\pi_{t-s}\mathbf{Tot}_{s+1}\Pi X \to \pi_{t-s}\mathbf{Tot}_s\Pi X) \to {}^{\mathbf{P}}\,\mathrm{D}_2^{s,t} = \pi_{t-s}\mathbf{Tot}_\infty\Pi X(t).$$

Let $[x] \in \mathrm{D}_2^{s,t}$ be represented by $x : S^{t-s} \to \mathbf{Tot}_{s+1}\Pi X \to \mathbf{Tot}_s\Pi X$. By adjunction, view this as

$$x : \Delta[s]_+ \wedge S^{t-s} \to \Delta[s+1]_+ \wedge S^{t-s} \to \Pi X.$$

To create the morphism, one must "lift" this to a morphism $\phi(x) : \Delta_+ \wedge S^{t-s} \to \Pi X(t)$. The morphism x consists of a compatible collection of morphisms

$$x^{n,i_*} : \Delta^n[s]_+ \wedge S^{t-s} \to \Delta^n[s+1]_+ \wedge S^{t-s} \to X^{i_n},$$

one for each $i_* \in N\Delta_n$. The key point underlying the details below is that $X^{i_n}(t)$ is a Kan complex and also has trivial homotopy groups $\pi_k X^{i_n}(t)$ when $k > t$. At various points one needs to make choices to extend maps. These need to be compatible with the cosimplicial structure. At any given point, this will involve finitely many choices differing in simplicial degrees greater than t. Thus, it will always be possible to make the choices compatibly.

Define

$$\phi(x)^{0,i_*} : \Delta^0[r]_+ \wedge S^{t-s} \to X^{i_0}(t)$$

for all r and $i_* \in N\Delta_0$ as the composition

$$\Delta^0_+ \wedge S^{t-s} = \Delta^0[s+1]_+ \wedge S^{t-s} \xrightarrow{x^{0,i_*}} X^{i_0} \to X^{i_0}(t).$$

Since the coskeleton map is a functor, this definition is functorial.

Now, suppose that

$$\phi(x)^* : \Delta^*_+ \wedge S^{t-s} \to \Pi^* X(t),$$

is defined for $0 \le * \le k$, compatible with all coface and codegeneracy maps in this range. If $i_* \in N\Delta_{k+1}$ is degenerate, then define

$$\phi(x)^{k+1,i_*} : \Delta^{k+1}_+ \wedge S^{t-s} \to X^{i_{k+1}}(t)$$

by forcing the diagram

$$\begin{array}{ccc} \Delta^{k+1}_+ \wedge S^{t-s} & \xrightarrow{\phi(x)^{k+1,i_*}} & X^{i_{k+1}}(t) \\ {\scriptstyle s^j}\downarrow & & \| \\ \Delta^k_+ \wedge S^{t-s} & \xrightarrow{\phi(x)^{k,i'_*}} & X^{i_{k+1}}(t) \end{array}$$

to be commutative for every j such that $s_j(i'_*) = i_*$ for some $i'_* \in N\Delta_k$. Making these simultaneously commutative for all choices of j is possible because of the simplicial relations $s_i \circ s_j = s_{j+1} \circ s_i$ for $i \leq j$.

If $i_* \in N\Delta_{k+1}$ is not degenerate, then the cosimplicial structure requires that the diagrams

$$\begin{array}{ccc} \Delta^k_+ \wedge S^{t-s} & \xrightarrow{\phi(x)^{k,\partial_j(i_*)}} & X^{i_{k+1}}(t) \\ {\scriptstyle \partial^j}\downarrow & & \| \\ \Delta^{k+1}_+ \wedge S^{t-s} & \xrightarrow{\phi(x)^{k+1,i_*}} & X^{i_{k+1}}(t) \end{array}$$

for $0 \leq j < k$ and

$$\begin{array}{ccc} \Delta^k_+ \wedge S^{t-s} & \xrightarrow{\phi(x)^{k,\partial_k(i_*)}} & X^{i_k}(t) \\ {\scriptstyle \partial^k}\downarrow & & \downarrow{\scriptstyle X(i_{k+1}\to i_k)} \\ \Delta^{k+1}_+ \wedge S^{t-s} & \xrightarrow{\phi(x)^{k+1,i_*}} & X^{i_{k+1}}(t) \end{array}$$

be commutative. In other words, the map $\phi(x)^{k+1,i_*}$ is already determined on $\partial\Delta^{k+1}_+ \wedge S^{t-s}$.

Thus, to make the inductive step, one must fill in the dashed line so that the diagram

$$\begin{array}{ccc} \Delta^{k+1}[s+1]_+ \wedge S^{t-s} & \xrightarrow{x^{k+1,i_*}} & X^{i_{k+1}} \\ \downarrow & & \searrow \\ \Delta^{k+1}_+ \wedge S^{t-s} & \overset{\phi(x)^{k+1,i_*}}{\dashrightarrow} & X^{i_{k+1}}(t) \\ \uparrow & \nearrow & \\ \partial\Delta^{k+1}_+ \wedge S^{t-s} & & \end{array}$$

is commutative. If $s \geq k$, then $\Delta^{k+1}[s+1]_+ = \Delta^{k+1}_+$, so there is nothing to do. If $s < k$, then $\Delta^{k+1}[s+1]_+ \wedge S^{t-s} \subseteq \partial\Delta^{k+1}_+ \wedge S^{t-s}$, and the outer square commutes by induction. Therefore, it suffices for the dashed arrow to commute in the bottom triangle. As $s < k$, the arrow $\partial\Delta^{k+1}_+ \wedge S^{t-s} \to X^{i_{k+1}}$ corresponds to a map $S^{k+t-s+1} \to X^{i_{k+1}}(t)$. But, $k+t-s+1 > t$. Hence, $\pi_{k+t-s+1}X^{i_{k+1}}(t) = 0$. As $X(t)$ is a Kan complex, such a fill exists (see [16, page 35]).

Choose a fill for all choices of i_*. This completes the induction, giving

$$\phi(x)^* : \Delta^*_+ \wedge S^{t-s} \to \Pi^* X(t),$$

for $0 \leq * \leq k+1$. The process outlined earlier in the proof gives a base case. So, induction provides the desired map

$$\phi(x) : \Delta_+ \wedge S^{t-s} \to \Pi^* X(t).$$

If y is another morphism $S^{t-s} \to \mathbf{Tot}_{s+1}\Pi X$ representing the class $[x]$, then it is straightforward, using a similar inductive argument, to lift a homotopy between x and y to a homotopy between $\phi(x)$ and $\phi(y)$ so that the map is well-defined on the level of homotopy groups. Thus, the construction above determines a well-defined the map $\phi :^{\mathbf{HL}} \mathrm{D}_2^{s,t} \to {}^{\mathbf{P}}\,\mathrm{D}_2^{s,t}$.

It is easy to see that ϕ commutes with i. Indeed, on ${}^{\mathbf{HL}}\,\mathrm{D}_2$, i is given by restriction from $\Delta[s]$ to $\Delta[s-1]$. On ${}^{\mathbf{P}}\,\mathrm{D}_2$, i is given by mapping $X(t) \to X(t-1)$. Now, composing $\phi(x)$ with $\Pi X(t)$ gives a map that solves the lifting problem to define $\phi(i(x))$:

$$\begin{array}{ccc}
\Delta[s-1]_+ \wedge S^{t-s} & \xrightarrow{i(x)} & \Pi X \\
\downarrow & & \| \\
\Delta[s]_+ \wedge S^{t-s} & \xrightarrow{x} & \Pi X \\
\downarrow & & \| \\
\Delta[s+1]_+ \wedge S^{t-s} & \longrightarrow & \Pi X \\
\downarrow & & \downarrow \\
\Delta_+ \wedge S^{t-s} & \xrightarrow{\phi(x)} & \Pi X(t) \\
\| & & \downarrow \\
\Delta_+ \wedge S^{t-s} & \xrightarrow{\phi(i(x))} & \Pi X(t-1).
\end{array}$$

Therefore, ϕ commutes with i.

Extend ϕ to ${}^{\mathbf{HL}}\,\mathrm{E}_2$ by lifting

$$x : S^{t-s} \to F(s)$$

to

$$x : S^{t-s} \to \mathbf{Tot}_{s+1}\Pi X$$

and applying ϕ (recall the description in Section 4.1 of E_2). In other words, apply k and then ϕ. This defines an element of ${}^{\mathbf{P}}D_2^{s,t}$, and, by construction, $\phi(x)$ actually factors through the fiber $G(t)$. Indeed, since

$$x : \Delta[s+1] \wedge S^{t-s} \to \Pi X$$

comes from the fiber F_s, it restricts to the trivial map on

$$\Delta[s-1] \wedge S^{t-s}.$$

Therefore, x is trivial on the $t-1$-skeleton. Extending this to a map

$$x : \Delta \wedge S^{t-s} \to \Pi X(t)$$

as above does not change this, so that when one composes with $\Pi X(t) \to \Pi X(t-1)$, the map is homotopic to the constant map on the basepoint. To check that this determines a well-defined map on E_2-terms, one must check that if

$$x = y + j(z),$$

where $z \in \ker(\pi_{t-s+1}X_{s-1} \to \pi_{t-s+1}X_{s-2})$ and

$$x, y \in \ker(\pi_{t-s}F(s) \to \pi_{t-s}X_s/i(\pi_{t-s}X_{s+1})),$$

then $\phi(x) = \phi(z)$. But, since

$$\phi(j(z)) = \phi(k(j(z))) = \phi(0),$$

it follows that this definition of ϕ on ${}^{\mathbf{HL}}\,\mathrm{E}_2$ is indeed well-defined.

It remains only to check that ϕ respects j. Again, let $[x] \in \pi_{t-s}\mathbf{Tot}_s\Pi X$ be represented by

$$x : \Delta[s+1]_+ \wedge S^{t-s} \to \Pi X.$$

Recall that $j([x])$ is obtained by the map

$$\pi_{t-s}\mathbf{Tot}_{s+1}\Pi X \to \pi_{t-s-1}F(s+2),$$

given by lifting a horn

$$\Delta[s+1]_+ \wedge \Lambda^{s-t}_{i,+} \to \Delta[s+1]_+ \wedge S^{s-t} \to \Pi X$$

to

$$\Delta[s+2]_+ \wedge \Delta^{s-t}_+ \to \Pi X$$

and then restricting to the ith face to obtain

$$\Delta[s+2]_+ \wedge S^{t-s-1} \to \Pi X,$$

which by adjunction is in $\pi_{t-s-1}F(s+2)$. Lift

$$\Delta[s+2]_+ \wedge \Delta^{s-t}_+ \to \Pi X$$

as above to a map

$$\Delta_+ \wedge \Delta_+^{s-t} \to \Pi X(t+1).$$

By construction, it maps down to

$$\phi(x) : \Delta_+ \wedge S^{t-s} \to \Pi X(t),$$

while the restriction to $\Delta_+ \wedge S^{t-s-1}$ is in the fiber $G(t+1)$. Therefore, ϕ commutes with j.

To prove that ϕ is injective on E_2, let $x \in {}^{\mathbf{HL}}\,\mathrm{E}_2^{s,t}$ be represented by

$$x : \Delta[s+1]_+ \wedge S^{t-s} \to \Pi X,$$

and let

$$\phi(x) : \Delta_+ \wedge S^{t-s} \to \Pi X(t)$$

factor through $G(t)$. Suppose that $\phi(x)$ is homotopic to the constant map in $G(t)$, and let

$$y : \Delta_+^1 \wedge \Delta_+ \wedge S^{t-s} \to G(t) \to \Pi X(t)$$

be such a homotopy. It is possible to find an extension

$$\begin{array}{ccc} \Delta_+^1 \wedge \Delta[s]_+ \wedge S^{t-s} & \dashrightarrow & \Pi X \\ \downarrow & & \downarrow \\ \Delta_+^1 \wedge \Delta_+ \wedge S^{t-s} & \xrightarrow{y} & \Pi X(t) \end{array}$$

giving a homotopy between x and the constant map by definition of the coskeleton $\Pi X(t)$. Therefore, ϕ is injective on E_2.

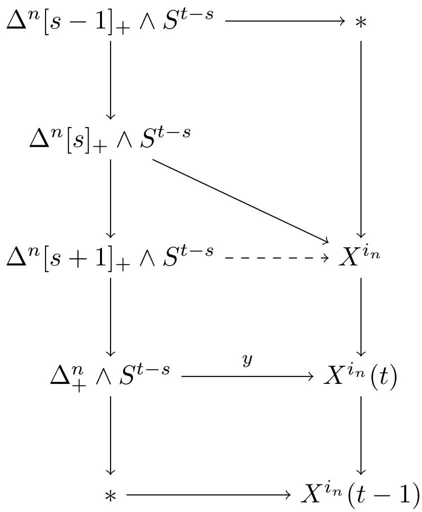
To show that ϕ is surjective on E_2, let

$$y : S^{t-s} \to G(t)$$

be represented by

$$\begin{array}{ccc} \Delta_+ \wedge S^{t-s} & \xrightarrow{y} & \Pi X(t) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Pi X(t-1). \end{array}$$

A lift for the diagram



exists by definition for $n \leq s$. For $n > s$, the top square means that the map

$$\Delta^n[s]_+ \wedge S^{t-s} \to X^{i_n}$$

is a diagram of t-spheres in X^{i_n}, and we want to trivialize the diagram. But, the diagram says that this diagram is trivialized in $X^{i_n}(t)$. Once again, by definition of the coskeleton functor, it follows that it is already trivializable in X^{i_n}. Now, arguing as above, one may inductively choose extensions in the diagram to create an element x of ${}^{\mathbf{HL}}\,\mathrm{E}_2^{s,t}$ such that $\phi(x) = y$. Therefore, ϕ is also surjective on E_2.

□

Remark 4.5.2. This theorem does not appear to be new. Indeed, it appears that Thomason was aware of it: see [29, page 542]. However, I know of no reference for a proof. It in fact holds in the greater generality of small diagrams of simplicial sets.

Corollary 4.5.3. *Suppose that X is a level Kan cosimplicial pointed space. If one of the spectral sequences*

$$ {}^{\mathbf{HL}}\,\mathrm{E}_1\, X \rightharpoonup \pi_{t-s}\operatorname{holim}_\Delta X$$
$$ {}^{\mathbf{P}}\,\mathrm{E}_2\, X \rightharpoonup \pi_{t-s}\operatorname{holim}_\Delta X.$$

converges, then both do, and the filtrations ${}^{\mathbf{HL}}Q_s$ and ${}^{\mathbf{P}}Q_s$ coincide on $\pi_* \operatorname{holim}_\Delta X$.

4.6 Closed Model Structure on Simplicial Presheaves

Let C be a Grothendieck site with terminal object U. Denote by $\mathbf{Pre}(C)$ and $\mathbf{Shv}(C)$ the categories of presheaves and sheaves on C, and write $\mathbf{sPre}(C)$ and $\mathbf{sShv}(C)$ for the categories of simplicial presheaves and simplicial sheaves.

I use the following closed model category structure on simplicial presheaves. The cofibrations are the pointwise cofibrations. Thus, $X \to Y$ is a cofibration if and only if $X(V) \to Y(V)$ is a monomorphism for every object V of C. For an object V of C, there is a site with terminal object $C_{/V}$. Each presheaf or sheaf on C restricts to a presheaf or sheaf on $C_{/V}$. For a simplicial presheaf X, an object V of C, and a basepoint $x \in X(V)_0$, there are presheaves of homotopy groups $\pi_k^p(X|V, x)$:

$$(f : W \to V) \mapsto \pi_k(|X(W)|, f^*(x)),$$

where $|X(W)|$ denotes the geometric realization of the simplicial set $X(W)$. Let $\pi_k(X|V, x)$ be the associated homotopy sheaf. Call $w : X \to Y$ a weak equivalence if it induces an isomorphism of homotopy sheaves

$$\pi_k(X|V, x) \xrightarrow{\cong} \pi_k(Y|V, w(x))$$

for all choices of V, all basepoints x of $X(V)$, and all $k \geq 0$. The fibrations are all maps having the right lifting property with respect to all cofibrations that are simultaneously weak equivalences (the acyclic cofibrations). That this is a simplicial closed model category is due to Joyal; for a proof, see [19]. Refer to these classes of morphisms more specifically as global fibrations, global cofibrations, and local weak equivalences.

Theorem 4.6.1 ([12]). *If X is a simplicial presheaf, and if $\mathcal{U}^\bullet \to U$ is a hypercover, then let $X_{\mathcal{U}^\bullet}$ denote the cosimplicial space associated to $\mathcal{U}^\bullet$. There is a canonical augmentation $X(U) \to X_{\mathcal{U}^\bullet}$. The simplicial presheaf X is globally fibrant if and only if*

$$X(U) \to \operatorname{holim}_\Delta X_{\mathcal{U}^\bullet}$$

is a weak equivalence for all hypercovers $\mathcal{U}^\bullet \to U$.

There are other types of morphisms, namely pointwise weak equivalences and pointwise fibrations. A pointwise weak equivalence is a morphism $f : X \to Y$ such that $X(V) \xrightarrow{\sim} Y(V)$ is a weak equivalence of simplicial sets for all objects V of C. Two pointwise weak equivalent sheaves are local weak equivalent, and two local weak equivalent fibrant presheaves are pointwise weak equivalent. A pointwise fibration is a morphism $f : X \to Y$ such that every $f : X(V) \twoheadrightarrow Y(V)$ is a fibration of simplicial sets. Say that X is pointwise Kan or pointwise fibrant if $X \to *$ is a pointwise fibration

Note that if a simplicial presheaf is pointed, then the homotopy presheaves and sheaves above may be defined globally.

Let F be a functor from simplicial sets to simplicial sets such that $F(\emptyset) = \emptyset$, or from pointed simplicial sets to pointed simplicial sets such that $F(*) = *$. If X is a simplicial presheaf, denote by FX the pointwise application of F to X, so that $(FX)(V) = F(X(V))$ for all V. For instance, below, $\operatorname{cosk}_n X$ will be the pointwise n-coskeleton of X. If F preserves weak equivalences of simplicial sets, then it preserves local weak equivalences of simplicial presheaves. This is the case, for instance, for the coskeleta functors and for the Ex functor. In particular, one may always replace X with the pointwise weakly equivalent presheaf $\operatorname{Ex}^\infty X$, which is pointwise Kan.

4.7 Spectral Sequences for Presheaves of Spaces

Definition 4.7.1. If X is a presheaf of pointed spaces, and if $X \to \mathbb{H}X$ is a pointed fibrant resolution of X, then $\mathbb{H}X$ is called the *hypercohomology spectrum* of X. Hypercohomology sets, groups, and abelian groups are defined by

$$\mathbb{H}^s(U, X) = \pi_s \Gamma(U, \mathbb{H}X).$$

Definition 4.7.2. Let X be a presheaf of pointed simplicial sets on $C \downarrow U$, and let $X \to \mathbb{H}X$ be a pointed fibrant resolution. Finally, let $X(n)$ be a pointed fibrant resolution of $\operatorname{cosk}_n \mathbb{H}X$ so that $X(n) \to X(n-1)$ is a fibration for all $n \geq 0$. Then, **the Brown-Gersten spectral sequence associated to** X is the $\tilde{\mathrm{E}}_2$-spectral sequence associated to the tower of fibrations

$$\Gamma(U, X(n)) \to \Gamma(U, X(n-1)).$$

That is, define the Brown-Gersten spectral sequence as

$$^{\mathbf{BG}}\,\mathrm{E}_2^{s,t}\, X = \tilde{\mathrm{E}}_2^{s,t}\{\Gamma(U, X(*))\} \rightharpoonup \pi_{t-s} \lim_{\leftarrow} \Gamma(U, X(n)).$$

Lemma 4.7.3 ([14])**.** *Suppose that the cite C is locally of finite cohomological dimension. Then, the natural morphism*

$$\Gamma(U, \mathbb{H}X) \to \lim_{\leftarrow} \Gamma(U, X(n))$$

is a weak equivalence for all objects U whenever X is locally weak equivalent to a product of a constant pointed space and a connected space.

In the situation of the lemma, write the Brown-Gersten spectral sequence as

$$^{\mathbf{BG}}\,\mathrm{E}_2\, X \rightharpoonup \mathbb{H}^{t-s}(U, X).$$

For instance, the lemma holds for presheaves of K-theory spaces on the étale sites of schemes of finite étale cohomological dimension.

The theorem of Dugger-Holander-Isaksen, Theorem 4.6.1, says that the natural map

$$\Gamma(U, X(n)) \to \mathbf{Tot}_\infty \Pi X(n)_{\mathcal{U}^\bullet}$$

is an isomorphism whenever $\mathcal{U}^\bullet \to U$ is a hypercover, and $X(n)_{\mathcal{U}^\bullet}$ denotes the cosimplicial space obtained by evaluating $X(n)$ at each level of $\mathcal{U}^\bullet$.

Definition 4.7.4. Let $\mathcal{U}^\bullet$ be a hypercover of an object U of C. Let X be a simplicial presheaf. Then, let $X_{\mathcal{U}^\bullet}$ denote the cosimplicial space one gets by evaluating X at $\mathcal{U}^\bullet$. The Čech hypercohomology of X on $\mathcal{U}^\bullet$ is defined as

$$\check{\mathbb{H}}^s(\mathcal{U}^\bullet, X) = \pi_s \mathbf{Tot}_\infty \mathbb{H}_c X_{\mathcal{U}^\bullet}.$$

Definition 4.7.5. The Čech hypercohomology spectral sequence associated to X and $\mathcal{U}^\bullet \to U$ is the total space spectral sequence associated to $\mathbb{H}_c X_{\mathcal{U}^\bullet}$:

$${}^{\mathcal{U}^\bullet}\mathrm{E}_1\, X = {}^{\mathbf{T}}\mathrm{E}_1\, X_{\mathcal{U}^\bullet}.$$

By Section 4.3, the E_2-terms for the corresponding spectral sequence are naturally identified with

$$\mathrm{E}_2^{s,t} \simeq \check{\mathrm{H}}^s(\mathcal{U}^\bullet, \pi_t^p X),$$

as desired.

There is a natural morphism from Čech hypercohomology to hypercohomology induced by a natural morphism

$$\mathbf{Tot}_\infty \mathbb{H}_c X_{\mathcal{U}^\bullet} \to \Gamma(U, \mathbb{H}X). \tag{4.6}$$

This morphism is the composition of

$$\mathbf{Tot}_\infty \mathbb{H}_c X_{\mathcal{U}^\bullet} \to \mathbf{Tot}_\infty \mathbb{H}_c \Pi^* X_{\mathcal{U}^\bullet} = \mathbf{Tot}_\infty \Pi^* X_{\mathcal{U}^\bullet} \to \mathbf{Tot}_\infty \Pi^* (\mathbb{H}X)_{\mathcal{U}^\bullet}$$

with the inverse of the local weak equivalence (Theorem 4.6.1)

$$\Gamma(U, \mathbb{H}X) \xrightarrow{\sim} \mathbf{Tot}_\infty \Pi^* (\mathbb{H}X)_{\mathcal{U}^\bullet}.$$

4.8 The Morphism of Spectral Sequences

Theorem 4.8.1. *Let X be a pointed simplicial presheaf, and let $\mathcal{U}^\bullet \to U$ be a hypercover. There is a morphism*

$${}^{\mathcal{U}^\bullet}\mathrm{E}_2\, X \to {}^{\mathbf{BG}}\mathrm{E}_2\, X$$

from the Čech hypercohomology spectral sequence to the Brown-Gersten spectral sequence, which on E_2-terms is the natural map

$$\check{\mathrm{H}}^s(\mathcal{U}^\bullet, \pi_t^p X) \to \mathrm{H}^s(U, \pi_t X).$$

This morphism respects the morphism on abutments of Equation (4.6).

Proof. One may assume, possibly by applying the Ex^∞ that X is pointwise Kan. Therefore, $X_{\mathcal{U}^\bullet}$ is a level Kan cosimplicial pointed space. Then, the space $\Pi X_{\mathcal{U}^\bullet}$ is fibrant, so that there is a natural morphism

$$\mathbb{H}_c X_{\mathcal{U}^\bullet} \to \Pi X_{\mathcal{U}^\bullet}$$

making the diagram

$$\begin{array}{ccc} X_{\mathcal{U}^\bullet} & \longrightarrow & \Pi X_{\mathcal{U}^\bullet} \\ & \searrow \quad \nearrow & \\ & \mathbb{H}_c X_{\mathcal{U}^\bullet} & \end{array}$$

commutative. This induces a morphism of total space spectral sequences

$$^{\mathcal{U}^\bullet}\,\mathrm{E}_2\,X \to {}^{\mathbf{T}}\,\mathrm{E}_2\,\mathbb{H}_c X_{\mathcal{U}^\bullet} \to {}^{\mathbf{HL}}\,\mathrm{E}_2\,X_{\mathcal{U}^\bullet} \tag{4.7}$$

There is a morphism

$$^{\mathbf{HL}}\,\mathrm{E}_2\,X_{\mathcal{U}^\bullet} \to {}^{\mathbf{HL}}\,\mathrm{E}_2(\mathbb{H}X)_{\mathcal{U}^\bullet} \tag{4.8}$$

induced by $X \to \mathbb{H}X$. Theorem 4.5.1 furnishes an isomorphism

$$^{\mathbf{HL}}\,\mathrm{E}_2(\mathbb{H}X)_{\mathcal{U}^\bullet} \xrightarrow{\simeq} {}^{\mathbf{P}}\,\mathrm{E}_2(\mathbb{H}X)_{\mathcal{U}^\bullet}. \tag{4.9}$$

Again, the fibrant replacement $\mathrm{cosk}_n\,\mathbb{H}X \to X(n)$ induces a morphism

$$^{\mathbf{P}}\,\mathrm{E}_2(\mathbb{H}X)_{\mathcal{U}^\bullet} \to \tilde{\mathrm{E}}_2^{s,t}\{\mathbf{Tot}_\infty \Pi X(n)_{\mathcal{U}^\bullet}\}. \tag{4.10}$$

Finally, the result of Dugger-Hollander-Isaksen [12] says that the natural morphism

$$^{\mathbf{BG}}\,\mathrm{E}_2\,X \to \tilde{\mathrm{E}}_2^{s,t}\{\mathbf{Tot}_\infty \Pi X(n)_{\mathcal{U}^\bullet}\} \tag{4.11}$$

is in fact an isomorphism.

The theorem follows by taking the composition of the morphisms of Equations (4.7), (4.8), (4.9), and (4.10), and the inverse of the morphism of Equation (4.11). □

4.9 Computation of the Fibers of the Total Space Tower

Take X to be a fibrant pointed cosimplicial space. I explain the isomorphisms $F(s) \simeq \mathbf{Map}_*(S^s, NX^s)$, where $F(s)$ is the fiber $\mathbf{Tot}_s X \to \mathbf{Tot}_{s-1} X$ and NX^s is the fiber of the fibration $X^s \to M^{s-1}X$. Let $\beta : \Delta^n_+ \to F(s)$. By adjunction, this is a morphism

$\Delta[s]_+ \wedge \Delta^n_+ \xrightarrow{\beta} X$ such that the restriction to $\Delta[s-1]_+ \wedge \Delta^n_+$ factors through the basepoint. In particular, the level s picture is $\Delta^s_+ \wedge \Delta^n_+ \to X^s$ such that the restriction to $\Delta^s[s-1]_+ \wedge \Delta^n_+$ is $*$. Therefore, this is a morphism $S^s \wedge \Delta^n_+ \to X$. That is, β determines an n-cell of $\mathbf{Map}_*(S^s, X^s)$. However, the degeneracy $s^i\beta : \Delta^{s-1}[s]_+ \wedge \Delta^n_+ \to X^{s-1}$ factors through $\Delta^{s-1}[s-1]_+ \wedge \Delta^n_+$ and so is trivial. Therefore the n-cell of $\mathbf{Map}_*(S^s, X^s)$ actually lives in $\mathbf{Map}_*(S^s, NX^s)$. Conversely, suppose that γ is an n-cell of $\mathbf{Map}_*(S^s, NX^s)$. Again, by adjunction, view this as a map $S^s \wedge \Delta^n_+ \to NX^s \to X^s$. Extend this as follows into a map $\Delta[s]_+ \wedge \Delta^n_+ \to X$. The lift of γ to $\Delta^s_+ \wedge \Delta^n_+$ is the s-level of this morphism. Since λ factors through $*$ for any degeneracy on Δ^s, let $\Delta^k[s]_+ \wedge \Delta^n_+ \to * \to X^k$ for $k < s$. If $k > s$, use the following diagram:

$$\begin{array}{ccc}
\Delta^s[s]_+ \wedge \Delta^n_+ & \xrightarrow{\beta} & X^s \\
\partial \big\downarrow & & \partial \big\downarrow \\
\Delta^k[s]_+ \wedge \Delta^n_+ & \xrightarrow{\partial\circ\beta\circ s} & X^k \\
s \big\downarrow & & s \big\downarrow \\
\Delta^s[s]_+ \wedge \Delta^n_+ & \xrightarrow{\beta} & X^s,
\end{array}$$

where the face map ∂ (resp. the degeneracy map s) is some composition of face (resp. degeneracy) maps such that the vertical compositions are the identity. This defines the extension inductively on the faces of $\Delta^k[s]$. But, one need only define it up to faces, since $k > s$. This gives us an n-cell of $\mathbf{Tot}_s X$. Clearly, these constructions are mutually inverse.

Now, it is not hard, using this identification, to show that the homotopy groups of $F(s)$ are the groups of the normalized cochain complexes associated to X. That is

$$\pi_{t-s}F(s) \simeq \pi_{t-s}\mathbf{Map}_*(S^s, NX^s) \simeq \pi_t X^s \cap \ker s^0 \cap \cdots \cap \ker s^{s-1},$$

where $\ker s^i$ is $\ker(s^i : \pi_t X^s \to \pi_t X^{s-1}$. See [5, proposition X.6.3].

4.10 Computation of d_1

I briefly indicate how to show that d_1 is homotopic to the cochain differential on the normalized cochain complex for $\pi_t X$. Let β this time represent a class in $\pi_{t-s}\mathbf{Map}_*(S^s, NX^s)$. The method above gives us a corresponding map $S^s \wedge S^{t-s} \to X^s$. Extending this, one gets a map $\Delta[s]_+ \wedge S^{t-s} \to X$. Lift this, using some horn Λ^{t-s}_i, to a map $\Delta[s+1]_+ \wedge \Delta^{t-s}_+ \to X$. On $\Delta[s]_+ \wedge \Delta^{t-s}_+$ this agrees with β. Restricting to the ith face of Δ^{t-s}, one gets a map $\Delta[s+1]_+ \wedge S^{t-s-1} \to X$. Then, look at just $\Delta^{s+1}_+ \wedge S^{t-s-1} \to X^{s+1}$, and check that one lands in the fiber. The reason that this agrees with the alternating sum $d = \Sigma(-1)^i d^i$ is that the cells $\Delta^{s+1}_+ \wedge \Delta^{t-s}_+ \to X^{s+1}$ give a homotopy between $d(\beta)$ and $d_1(\beta)$.

4.11 Computation of d_2

Let $\delta : S^t \to X^s$ represent a class $[\delta]$ of $\pi^s\pi_t X$, where $t - s \geq 0$, and suppose that δ factors as $\delta : S^{t-s} \to \mathbf{Map}_*(S^s, NX^s)$. That is, suppose that δ represents $[\delta]$ in the normalized chain complex. Then, as above, extended δ to a map

$$\delta' : \Delta[s]_+ \wedge \Delta_+^{t-s} \to X.$$

Now, the restriction of this map to $\Delta[s]_+ \wedge \partial\Delta_+^{t-s}$ factors through the base-point. Choosing a horn $\Lambda_i^{t-s} \subset \partial\Delta^{t-s}$, one gets a lift to a map

$$\gamma : \Delta[s+1]_+ \wedge \Delta_+^{t-s} \to X$$

such that the following diagram commutes

$$\begin{array}{ccc} \Delta[s+1]_+ \wedge \Delta_+^{t-s} & \xrightarrow{\gamma} & X \\ \uparrow & & \| \\ \Delta[s]_+ \wedge \Delta_+^{t-s} & \xrightarrow{\delta'} & X. \end{array}$$

The fact that δ represents a cohomotopy class implies, by Section 4.10, that the restriction of γ to

$$\Delta_+^{s+1} \wedge \partial_i \Delta_+^{t-s} \to X^{s+1}$$

is contractible in $\mathbf{Map}_*(S^{s+1}, NX^{s+1})$. Therefore, one can replace γ by a homotopic map γ' such that the restriction to $\gamma' : \Delta[s+1]_+ \wedge \partial\Delta_+^{t-s} \to X$ factors through the base-point. Thus, one can repeat the process to get a lift $\epsilon : \Delta[s+2]_+ \wedge \Delta_+^{t-s} \to X$, using some horn Λ_j^{t-s}. The differential $d_2([\delta])$ is the class of $\epsilon : \Delta_+^{s+2} \wedge \partial_j \Delta_+^{t-s} \to X^{s+2}$ in $\pi^{s+2}\pi_{t-s-1}X$.

Note the following, which will be an aid to making the extensions described above. I claim that to extend $\Delta[s]_+ \wedge \Delta_+^{t-s} \to X$ to $\Delta[s+1]_+ \wedge \Delta_+^{t-s}$ it is sufficient to describe the extension of $\Delta^{s+1}[s]_+ \wedge \Delta_+^{t-s} \to X^{s+1}$ to $\Delta_+^{s+1} \wedge \Delta_+^{t-s}$. Indeed, make the same argument as in Section 4.9.

Now that I have described d_1 and d_2, it is easy to see how to extend this description to higher differentials d_i. Indeed, each time d_i vanishes, this means that one can perform one more lift as above.

4.12 Description of d_2 for the Čech Spectral Sequence

Now, let X be a presheaf of pointed simplicial sets on the site C, and let $\mathcal{V}_A \to \mathcal{U}_I \to U$ be a 1-hypercover in C. Thus, for $\alpha \in A$, $V_{ij}^{\alpha} \to U_{ij}$ is a covering morphism, where $U_{ij} = U_i \times_U U_j$.

I describe the Čech differential d_2 when $s = 0$ and $t \geq 0$. Let $[\delta] \in \pi^0\pi_t X_{\mathcal{U}^\bullet}$, with $t \geq 1$, be represented by

$$\delta : S^t \to X^0_{\mathcal{U}^\bullet} = \prod_{i \in I} X(U_i).$$

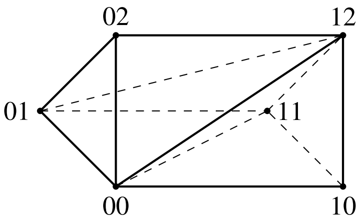
Following the recipe above, this extends to a map $\Delta[0]_+ \wedge \Delta^t \to X_{\mathcal{U}^\bullet}$ so that the composition

$$\Delta^k[0]_+ \wedge \Delta^t_+ \to X^k_{\mathcal{U}^\bullet}$$

is the map determined by the diagrams

$$\begin{array}{ccc} \Delta^0[0]_+ \wedge \Delta^t_+ & \xrightarrow{\delta} & X^0_{\mathcal{U}^\bullet} \\ \partial\big\downarrow & & \partial\big\downarrow \\ \Delta^k[0]_+ \wedge \Delta^t_+ & \xrightarrow{\partial\circ\delta\circ s} & X^k_{\mathcal{U}^\bullet} \\ s\big\downarrow & & s\big\downarrow \\ \Delta^0[0]_+ \wedge \Delta^t_+ & \xrightarrow{\delta} & X^0_{\mathcal{U}^\bullet}, \end{array}$$

one for each vertex of Δ^k. Now, use homotopies on the V^α_{ij} to extend this to a map $\Delta[1]_+ \wedge \Delta^t_+ \to X_{\mathcal{U}^\bullet}$. Write x_i for the projection of $\Delta^t_+ \to \prod_i X(U_i) \to X(U_i)$. Then, let y^α_{ij} denote an arbitrary homotopy $x_i|V^\alpha_{ij} \xrightarrow{\simeq} x_j|V^\alpha_{ij}$. These fit together to give an extension $\Delta[1]_+ \wedge \Delta^t_+ \to X_{\mathcal{U}^\bullet}$ as desired. In order to describe an extension $\Delta[2]_+ \wedge \Delta^t_+ \to X_{\mathcal{U}^\bullet}$, it is necessary and sufficient to fill in the diagrams



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where I now assume, for ease of display, that $t = 1$, and that the square $(00, 01, 11, 10)$ is the homotopy y^β_{kj}, the square $(01, 02, 12, 11)$ is the homotopy y^α_{ji}, and the square $(00, 02, 12, 10)$ is the homotopy y^δ_{ki}. Here, I assume that the 2-cell $(10, 11, 12)$ is $*$. The differential $d_2(\delta)$ is then the class of $(00, 01, 02)$ in $\pi^2\pi_2 X_{\mathcal{U}^\bullet}$.

4.13 Divisibility Theorem

Theorem 4.13.1. *Suppose that U is geometrically connected and quasi-separated. Let $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$. Then, $d^\alpha_2([1]) = \alpha$, through the natural isomorphism* $\mathrm{H}^2(U_{\text{ét}}, \mathcal{K}^\alpha_1) \xrightarrow{\simeq}$

$\mathrm{H}^2(U_{\text{ét}}, \mathcal{K}_1)$.

Proof. Let $\mathcal{V} \to \mathcal{U} \to U$ be a 1-hypercover that trivializes α, and write $\alpha_{\mathcal{U}^\bullet}$ for a Čech cocycle representing α on this cover. Let

$$Z_{ijk}^{\alpha\beta\delta} = V_{ij}^{\alpha} \times_{U_j} V_{jk}^{\beta} \times_{U_k} V_{ik}^{\delta}.$$

I may represent $[1] \in \check{\mathrm{H}}^0(\mathcal{U}^\bullet, K_0^\alpha)$ by an α-twisted line bundle $\mathcal{L}_i$ on each U_i of $\mathcal{U}^\bullet$. A homotopy from $\mathcal{L}_i$ to $\mathcal{L}_j$ is just an isomorphism $\theta_{ij}^\alpha : \mathcal{L}_i|_{V_{ij}^\alpha} \xrightarrow{\simeq} \mathcal{L}_j|_{V_{ij}^\alpha}$, where such an isomorphism exists, up to possibly refining the hypercover $\mathcal{U}^\bullet$, that such an isomorphism exists. Then, by Diagram 4.12, the class of the automorphism $\theta_{ki}^\delta \circ \theta_{jk}^\beta \circ \theta_{ij}^\alpha$ of $\mathcal{L}_i|_{Z_{ijk}^{\alpha\beta\delta}}$ in $\mathbf{K}_1^\alpha(U_{ijk}^{\alpha\beta\delta})$ is the $Z_{ijk}^{\alpha\beta\delta}$-component of $d_2^\alpha([1])$. The α-twisted line bundles $\mathcal{L}_i$ on U_i induce pointwise weak equivalences of K-theory presheaves $\phi_i : \mathbf{K}^\alpha|_{U_i} \xrightarrow{\simeq} \mathbf{K}|_{U_i}$ by tensor product with $\mathcal{L}_i^{-1}$. As shown in 2.10, these local morphisms patch to create natural isomorphisms of K-theory sheaves

$$\mathcal{K}_k^\alpha \xrightarrow{\simeq} \mathcal{K}_k,$$

and of K-cohomology groups, in particular of

$$\mathrm{H}^2(U_{\text{ét}}, \mathcal{K}_1^\alpha) \xrightarrow{\simeq} \mathrm{H}^2(U_{\text{ét}}, \mathcal{K}_1).$$

The rank 1 objects of $\mathbf{Proj}^\alpha$ form a $\mathbb{G}_m$-gerbe, $\mathbf{Pic}^\alpha$. Recall (from [15, Section IV.3.4] or [7, Theorem 5.2.8]) that one constructs a 2-cocycle in $\mathbb{G}_m$ from a $\mathbb{G}_m$-gerbe $\mathbb{G}$ by precisely the procedure above. First, one takes a cover $\mathcal{U}$ of U such that there is an object of $a_i \in \mathbb{G}|_{U_i}$ for all i. Second, choose isomorphisms

$$\sigma_i : \mathrm{Aut}(a_i) \xrightarrow{\simeq} \mathbb{G}_m|_{U_i},$$

Third, choose isomorphisms

$$\delta_{ij}^\alpha : a_i \xrightarrow{\simeq} a_j$$

on a suitable refinement 1-hypercover $\mathcal{V}_A \to \mathcal{U}_I \to U$. The composition

$$\delta_{ki}^\delta \circ \delta_{jk}^\beta \circ \delta_{ij}^\alpha$$

is an element of $\mathrm{Aut}(a_i)(Z_{ijk}^{\alpha\beta\delta})$, and

$$\sigma_i(\delta_{ki}^\delta \circ \delta_{jk}^\beta \circ \delta_{ij}^\alpha)$$

defines a 2-cocycle in $\mathbb{G}_m$ which is in the same cohomology class as α.

Let $a_i = \mathcal{L}_i$. Define σ_i by fixing an isomorphism

$$\sigma_i : \mathcal{L}_i \otimes \mathcal{L}_i^{-1} \xrightarrow{\simeq} \mathbb{G}_m|_{U_i},$$

possibly refining $\mathcal{U}^\bullet$. Then, the diagram

$$\begin{array}{ccc} \mathrm{Aut}(\mathcal{L}_i) & \xrightarrow{\sigma_{i,*}} & \mathbb{G}_m|_{U_i} \\ \downarrow & & \downarrow \\ \mathbf{K}_1^\alpha(U_i) & \xrightarrow{\phi_i} & \mathbf{K}_1(U_i) \end{array}$$

is commutative, where $\sigma_{i,*}$ is the natural isomorphism induced by σ_i. Setting $\delta_{ij}^\alpha = \theta_{ij}^\alpha$, the diagram implies that $d_2^\alpha([1])$ maps to the image of $\alpha_{\mathcal{U}^\bullet}$ in the map $\mathrm{H}^2(\mathcal{U}^\bullet, \mathbb{G}_m) \to \mathrm{H}^2(\mathcal{U}^\bullet, \mathbf{K}_1)$. As this remains true under refinements of the 1-hypercover, the theorem follows. □

Remark 4.13.2. The hypothesis of quasi-separatedness is required so that sheaf cohomology is computable with hypercovers. See [3, Theorem V.7.4.1].

Theorem 4.13.3 (Divisibility). *Let U be geometrically connected and quasi-separated. If $\alpha \in \mathrm{H}^2(U_{\text{ét}}, \mathbb{G}_m)$, then*

$$per(\alpha) | spi(\alpha).$$

Proof. This follows immediately from Theorem 4.13.1. □

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