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FRG Workshop.

Higher algebraic geometry 3.

$$\begin{aligned} \mathrm{Gest} &= \mathrm{CatRing}^{op} \\ &\;\;\, \| \\ &\left(\lim \mathrm{CAlg}(n\hat{\Pr})\right)^{op} \end{aligned}$$

$\mathcal{C}$ cat w/finite limits

$\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \longrightarrow \hat{\Pr}$ $\rightsquigarrow$ Will produce $\mathcal{C} \longrightarrow \mathrm{Gest}$ pullback preserving

6FF

$$X \longrightarrow [X]_{\mathrm{Sh}} = \mathrm{Spec}(\underline{\mathrm{Sh}}(X))$$

$$\nearrow$$

$$(\mathrm{Sh}(X), 2\mathrm{Sh}(X), 3\mathrm{Sh}(X), \dots)$$

Assume Sh is symmetric monoidal.

$$\begin{tikzcd}
\mathcal{P}(\mathrm{Corr}(\mathcal{C})) \arrow[r, "\mathrm{Sh}"] \arrow[d, "\times X"] & \Pr \arrow[dd] \\
\mathcal{P}(\mathrm{Corr}(\mathcal{C}_{/X})) \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \\
\mathcal{P}(\underline{\mathrm{Corr}}(\mathcal{C}_{/X})) \arrow[r] & (2\mathrm{Sh}(X), 3\mathrm{Sh}(X), \dots)
\end{tikzcd}$$

$(\mathrm{Corr}(\mathcal{C}), B\mathrm{Corr}(\mathcal{C}), B^2\mathrm{Corr}(\mathcal{C}), \dots)$

$(\mathrm{Corr}(\mathcal{C}), 2\mathrm{Corr}(\mathcal{C}), 3\mathrm{Corr}(\mathcal{C}), \dots)$

$\underline{\mathrm{Corr}}(\mathcal{C}_{/X})$ categorical spectrum of correspondences

$(\mathrm{Corr}(\mathcal{C}), 2\mathrm{Corr}(\mathcal{C}), 3\mathrm{Corr}(\mathcal{C}), \dots)$

$n\mathrm{Corr}(\mathcal{C})$ ob = ob $\mathcal{C}$

$$\mathrm{Hom}_{n\mathrm{Corr}(\mathcal{C})}(X, Y) = (n-1)\mathrm{Corr}({}_X\mathcal{C}_Y).$$

$$\mathrm{CatSp} \underset{\text{forget}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathrm{CatRing}$$

$$\begin{tikzcd}
 & W \arrow[dl] \arrow[dr] & \\
X & U \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\
 & W' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

2cell in 2Corr.

Shifted because of the map from Pr.

Concretely.

$n\,\mathrm{Sh}(X)$ generated by

$(n-1)\,\mathrm{Sh}(Y/X)$, $Y \in \mathcal{C}_{/X}$.

$$\mathrm{Hom}_{n\,\mathrm{Sh}(X)}\big((n-1)\,\mathrm{Sh}(Y/X),\ (n-1)\,\mathrm{Sh}(Y'/X)\big) \simeq (n-1)\,\mathrm{Sh}(Y \times_X Y').$$

$\underline{n=2}$. Presentable version of cat. of kernels.

(Everything is 1-étale over $[*]_{\mathrm{sh}}$, $*$ final obj. of $\mathcal{C}$.)

Various ways of fixing this when $\overset{\mathrm{sh}}{-}$ only lex symmetric monoidal.
For ex. on kernels to get ∞. Or use some enriched cat. theory.

Why does $X \longmapsto [X]_{\mathrm{sh}}$ preserve pullbacks?
This is really the point where we need to go all the way to ∞.

$\underline{\mathrm{Sh}}(Y \times_X Y') \simeq \underline{\mathrm{Sh}}(Y) \otimes_{\underline{\mathrm{Sh}}(X)} \underline{\mathrm{Sh}}(Y')$ in CatRing.

Not typically lambdawise.

$n\,\mathrm{Sh}(Y \times_X Y' | X) \simeq n\,\mathrm{Sh}(Y|X) \otimes n\,\mathrm{Sh}(Y'|X)$ $\underline{\text{in}}$ $\underline{(n+1)\,\mathrm{Sh}(X)}$.

over unit $n\,\mathrm{Sh}(X)$

Somehow different from doing this in $(n+1)\mathrm{Pr}$?

Moduli of invertible TQFTs.

R categorical ring, $(R_1, R_2, \dots)$.

$$nPic(R) = \left\{ \begin{array}{c} \otimes\text{-invertibles} \\ \text{in } R_n \end{array} \right\}.$$

$$\simeq \mathrm{Hom}_{\mathrm{CatRing}}\left(nP(\mathbb{S}),\ R\right)$$

$$nP(\mathbb{S}) = \mathrm{Fun}(\mathbb{S}, (n-1)\mathrm{Pr})$$

Duality, $X \in \mathrm{Gest}$

$$nPic\left(\underline{\mathrm{Sh}}(X)\right) = \mathrm{Hom}_{\mathrm{Gest}}\left(X,\ \mathrm{Spec}\left(nP(\mathbb{S})\right)\right)$$ families of invertible n-dim TQFTs.

$\Omega^{\infty}\mathbb{S}$

$$\mathrm{Spec}\left(nP(\mathbb{S})\right) = nTQFT_{inv}$$

Moduli of invertible n-dim TQFTs.

Rem. $\Omega\,(n+1)TQFT_{inv} \simeq nTQFT_{inv}$. These are 1-étale.

These assemble into a spectrum Gestalt GL_1.

$$\tau_{\geq 0}\left(GL_1[n]\right) \simeq nTQFT_{inv},\ n \geq 1.$$

Rem. $\tau_{\geq 0}(GL_1) = $ usual GL_1

$$\underset{\text{over } k}{\simeq} \mathrm{Spec}\left(k[\Omega^{\infty}\mathbb{S}]\right).$$

(Also works over Spec Ani.)

<u>Thm</u> (Scholze – Stefanich). Over $\mathbb{C}$, $n \geq 1$

$$\pi_{-n} \mathrm{Gl}_1 \simeq \pi_{-n} \mathbb{S} \quad \left(= \mathrm{Hom}(\pi_n \mathbb{S}, \mathbb{C}^\times)\right)$$

Conj from Ani – Gest.

<u>Rem.</u> $\underline{\mathrm{Sh}} : \mathrm{Gest}^{op} \longrightarrow \mathrm{CatRing}$ (Restatement.)

"cat ring obj. in Gestalten" $\rightsquigarrow$ $\underline{\mathbb{A}}^1$.

$\pi_0\, n\mathrm{Pic}(\underline{\mathbb{A}}^1) \cong \pi_n \mathbb{S}$. Universal target for TQFTs.

$\mathrm{Sh} = \mathrm{IndCoh}$ on schemes of finite type over $\mathbb{C}$. (6FF)

X smooth

$2\mathrm{IndCoh}(X) \supseteq 2\mathrm{QCoh}(X)$ $\rightsquigarrow$ 3d TQFT (2 dualizable)

$\mathcal{X}(*) = 2\mathrm{IndCoh}(X)$ (Version $2\mathrm{IndCoh}(X | \mathrm{Spec}\, \mathbb{C}) \in 3\mathrm{IndCoh}(\mathrm{Spec}\, \mathbb{C})$.)

$\mathcal{X}(S^1) = \mathrm{IndCoh}(\mathrm{Map}(|S^1|, X))$ Just relies on 6FF.

$\simeq \mathrm{QCoh}(T^*[2]X)$.

Koszul + HKR

Compatible with Rozansky–Witten theory of T^*X.

<u>Thm</u> (Ben-Zvi – Nadler – Stefanich). There exists a cat. ring $k(u)_{add}$, $|u| = -2$. Makes things 2-periodic.

Underlying ring is $k(u^{\pm 1})$.

V f.dim. vector space.

$$2\mathrm{IndCoh}(V) \otimes k(u)_{add} \simeq 2\mathrm{IndCoh}(V^\vee) \otimes k(u)_{add} \quad \text{(Fourier transform).}$$

Action of higher central ext. of symplectic group.

Geometrize.

X scheme.

$$X^! = \mathrm{Spec}\left(\mathcal{Z}\,\mathrm{IndCoh}(X)\right).$$

Internal hom, additive maps.

Cor of Thm + Cartier duality. $$(V^*)^! = \underline{\mathrm{Hom}}\left(V^!, B^2\mathbb{G}_m\right).$$

$$\mathbb{G}_a^! = \underline{\mathrm{Hom}}\left(\mathbb{G}_a^!, B^2\mathbb{G}_m\right)$$

$$1 \rightsquigarrow \left(\exp : \mathbb{G}_a^! \to B^2\mathbb{G}_m\right)$$

Def. The additive de Rham stack of $B\mathbb{G}_m$ is

$$(B\mathbb{G}_m)_{dR,add} = \mathrm{fib}\left(\exp : \mathbb{G}_a^! \to B^2\mathbb{G}_m\right).$$

Looks like

$$(B\mathbb{G}_m)_{dR} \hookleftarrow \bigcirc \longrightarrow (\mathbb{G}_a)_{dR}.$$

In progress, but confident.

In general:

$$\mathrm{Glo} = \left\{ \mathrm{Spec}\,A / \underset{\substack{\uparrow \\ \text{reductive}}}{G} \right\}$$

$\exists$ LEX functor $$(-)_{dR,add} : \mathrm{Glo} \longrightarrow \mathrm{Gest}_{/k(0)_{add}}.$$

Also multiplicative and elliptic.

Conj. $2\mathrm{Sh}\left((B\check{G})_{dR,\mathbb{A}^1/\mathbb{G}_m}\right)$ ... Langlands.