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FRG Workshop.

Higher algebraic geometry 2.

$$n\widehat{\Pr} = \left\{ \underset{\text{on all } d\text{-cells } 0\le d\le n}{\text{cocomplete}} \ \ n\text{-cats} \right\}$$

$$\mathcal{E} \longmapsto \mathrm{Mod}_{\mathcal{E}}$$

$$\mathrm{CAlg}(1\widehat{\Pr}) \rightleftarrows \mathrm{CAlg}(2\widehat{\Pr}) \rightleftarrows \mathrm{CAlg}(3\widehat{\Pr}) \ \cdots \ \rightleftarrows \mathrm{CatRing}.$$

$$\mathrm{End}_{\mathcal{D}}(\mathbb{1}) \longleftarrow\!\!\shortmid \ \mathcal{D}$$

Def. The category of Gestalten ("shapes") is $\mathrm{Gest} = \mathrm{CatRing}^{op}$.

$$\begin{tikzcd}
\mathrm{Aff} \arrow[r, phantom, "="] \arrow[d, hook] \arrow[dd, bend right, phantom] & \mathbb{E}_\infty\text{-alg} \arrow[d, hook] \\
\mathrm{Gest} \arrow[r, phantom, "="] & \mathrm{CatRing}^{op}
\end{tikzcd}$$

$\mathcal{P}(\mathrm{Aff}) \xrightarrow[\mathrm{LKE}]{\circledast} \mathrm{Gest}$

$$\mathrm{Spec}\, R \longleftarrow\!\!\shortmid \ R$$

$$X \longmapsto \underbrace{\underline{\mathrm{Sh}}(X)}_{\text{associated cat. ring.}} = (\underbrace{1\mathrm{Sh}(X)}_{\mathrm{Sh}(X)}, 2\mathrm{Sh}(X), \dots)$$

If $X \in \mathcal{P}(\mathrm{Aff})$, denote $[X]_{\mathrm{QCoh}}$ the image of X under $\circledast$.

$$n\,\underline{\mathrm{Sh}}([X]_{\mathrm{QCoh}}) = n\mathrm{QCoh}(X).$$

Factors through fppf topology. Also for descendable.
None of the ~~[illegible]~~ induced $\circledast$ functors are fully faithful.

Ex. X a qcqs scheme

$$[X]_{\mathrm{QCoh}} = \mathrm{Spec}(\mathrm{QCoh}(X)) \quad \text{"1-affineness".}$$

via $\mathrm{CAlg}(1\Pr) \longleftarrow \mathrm{CatRing}$.

In general, $\mathrm{QCoh}(X) \longrightarrow \underline{\mathrm{QCoh}}(X)$.

$$\mathrm{Hom}\left([X]_{\mathrm{QCoh}}, [Y]_{\mathrm{QCoh}} \right) \simeq \mathrm{Hom}\left(\mathrm{QCoh}(Y), \mathrm{QCoh}(X) \right)$$

$$\simeq \mathrm{Hom}(X, Y)$$

Tannaka duality

<u>Ex</u>. k field

$B\mathbb{G}_m$ 1-affine

$$\underset{\mathrm{Gest}_{/\mathrm{Spec}\, k}}{\pi_0 \mathrm{Hom}}\left(\mathrm{Spec}\, k, [B\mathbb{G}_m]_{\mathrm{QCoh}} \right) \underset{\text{1-affineness}}{=} \pi_0 \mathrm{Hom}\left(\mathrm{QCoh}(B\mathbb{G}_m), \mathrm{Mod}_k \right)$$

$= \mathbb{Z} \cdot \text{shear}$ ← really shear composed with

$\text{shear}\left(V(\ast) \right) = \bigoplus V_{(\ast)}[2\ast]$.

Note $\pi_0 \mathrm{Hom}(\mathrm{Spec}\, k, B\mathbb{G}_m) = \ast$!

So, no fully faithfulness.

What are the higher homotopy groups?

<u>Ex</u>. k alg. closed field.

$$\pi_0 \mathrm{Hom}\left(\mathrm{Spec}\, k, [B^2 \mathbb{Z}/m]_{\mathrm{QCoh}} \right) = \mathbb{Z}/m$$

↑ Comes all the way Ank.

Generator is a $\mathbb{Z}/m$-gerbe over $\mathrm{Spec}\, k$ in Gest.

$$\begin{tikzcd} \mathrm{Spec}\left(\mathrm{Mod}_k[\sqrt[m]{k[z]}] \right) \arrow[r] \arrow[d] & \ast \arrow[d] \\ \mathrm{Spec}\, k \arrow[r] & {[B^2 \mathbb{Z}/m]_{\mathrm{QCoh}}} \end{tikzcd}$$

Extract m^{th} root of $k[z]$.

Are Anima seen in Gestalten, i.e. 1-affine?

Def. $f: X \longrightarrow Y$ a map of Gestalten.

Say f is n-étale if for all $i \geq 1$

$(n+i)\mathrm{Sh}$[redacted]$(Y) \longrightarrow (n+i)\mathrm{Sh}(X)$

is i-fold left adjointable.

$\underline{i=1}$. left adjointable. (has a left adj)

$\underline{i>1}$. right adjointable and unit/counit are $(i-1)$-left adjointable.

Point is to exchange limits and colimits.

$$\mathrm{Gest}^{0\text{-ét}}_{/Y} \subseteq \mathrm{Gest}^{1\text{-ét}}_{/Y} \subseteq \mathrm{Gest}^{2\text{-ét}}_{/Y} \subseteq \cdots \qquad \mathrm{Gest}_{/Y}$$

does not converge

special, related to classical étale maps

fairly general

$[X]_{QCoh}$ 1-étale$/\mathrm{Spec}(\mathbb{S})$

$X \in P(\mathrm{Aff})$.

Closed under colimits, finite limits,

$\prod$ n-ét is $(n+1)$-étale.

So, can just work with N-étale for some N, like 100.

Thm (Scholze–Stefanich). $\mathrm{Gest}^{n\text{-ét}}_{/Y}$ is (up to presentability) a topos.
(infinitary pretopos)

Not from a site, as far as they know.

Canonical top. is related to descendable top.

In particular, $X \longrightarrow [X]_{QCoh}$ left exact.

Transmutation. Geom. Her out of 6FF.

$\mathcal{C}$ w/ finite limits $\qquad$ (Eg $\mathcal{C} = \mathrm{Aff}$)

$\mathrm{Corr}(\mathcal{C})^{\otimes}$ $\quad$ ob = ob($\mathcal{C}$)

$$\mathrm{Hom}_{\mathrm{Corr}(\mathcal{C})}(X, Y) = \left\{ \begin{tikzcd}[column sep=small] & W \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \right\}.$$

$X \otimes Y = X \times Y.$

Def. A 6-ff on $\mathcal{C}$ is a lax $\otimes$-functor $\mathrm{Corr}(\mathcal{C})^{\otimes} \xrightarrow{\mathrm{Sh}} \Pr^{(\wedge)}$.

Do we really get all 6 in this generality?

Claim/construction (S.-S.). For $\mathcal{C}$, Sh a 6ff.

There is a pullback-preserving functor

$$\mathcal{C} \longrightarrow \mathrm{Geom}$$
$$X \longmapsto [X]_{\mathrm{Sh}}$$
$$\mathrm{Sh}([X]_{\mathrm{Sh}}) = \mathrm{Sh}(X).$$

Just the first level.

⚠ Not $X \mapsto \mathrm{Spec}(\mathrm{Sh}(X))$.

This is not pullback-preserving.

Related to categorical Künneth formulas.

In general, this fixes the failure of cat. Kunneth formulas.

(Sometimes can go back.)

Assume. $\mathrm{Sh}: \mathrm{Corr}(\mathcal{C}) \longrightarrow \Pr$ symmetric monoidal.

$$\begin{tikzcd} \mathcal{P}(\mathrm{Corr}(\mathcal{C})) \arrow[r] \arrow[d] & \Pr \arrow[d] \\ \mathcal{P}(\mathrm{Corr}(\mathcal{C}_{/X})) \arrow[r] & 2\mathrm{Sh}(X) \end{tikzcd} \qquad \text{p.o. in } \mathrm{CAlg}(\widehat{\Pr}).$$

Concretely: $2\mathrm{Sh}(X)$ generated by $\mathrm{Sh}(Y/X)$, $Y \in \mathcal{C}_{/X}$.

$\mathrm{Hom}(\mathrm{Sh}(Y/X), \mathrm{Sh}(Y'/X)) = \mathrm{Sh}(Y \times_X Y').$