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16 March 2025.
FRG Workshop.

Higher algebraic geometry 1.

Plan:

1. Categorical background,
2. Category of Gestalten.
3. Applications. (TFT from different angles).

Joint w/ Scholze.

R (commutative) ring.

Mod_R stable (∞-)category of R-modules.
symmetric monoidal

$\mathrm{End}_{\mathrm{Mod}_R}(\mathbb{1}) \simeq R$

So, $\{\text{rings}\} \longleftrightarrow \left\{\begin{matrix}\text{s.mon. [redacted]} \\ \text{categories w/ colimits}\end{matrix}\right\}$

$R \longmapsto \mathrm{Mod}_R$

cocomplete cat enriched in cocomplete cats.

$\left\{\begin{matrix}\text{cocomplete} \\ \otimes\text{-cats}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{cocomplete} \\ \otimes\text{-2-cats}\end{matrix}\right\} \longleftrightarrow \cdots$

Study the limit...

S another ring.

$$\mathrm{Map}(R,S) \simeq \mathrm{Map}^{\otimes,L}(\mathrm{Mod}_R, \mathrm{Mod}_S)$$

$$R \to S \longmapsto - \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$$

Ex. k ring

$$\mathrm{QCoh}(B\mathbb{G}_{m,k}) = \mathrm{Rep}(\mathbb{G}_{m,k}).$$
$$\simeq \prod_{\mathbb{Z}} \mathrm{Mod}_k$$

Not gen. by unit, so not Mod_R.

Enriched categories w/colimits.

$\mathcal{C}$ symmetric monoidal category

κ cardinal uncountable regular

$\mathcal{C}$ should be κ-presentable.

- $\mathcal{C}$ has colimits
- $\mathcal{C}$ gen. by a small collection of κ-compact objects
- $\otimes$ comp. w/colimits in each variable
- $\mathcal{C}^\kappa \subseteq \mathcal{C}$ sub symmetric monoidal cat.

Ex. Mod_k.

Df. A κ-cocomplete $\mathcal{C}$-enriched category is a category $\mathcal{D}$ with κ-small colimits together with an action

$$\mathcal{C}^\kappa \times \mathcal{D} \longrightarrow \mathcal{D}$$

which preserves κ-small colimits in each variable.

Rm. $X, Y \in \mathcal{D}$, get a hom object $\underline{\mathrm{Hom}}(X,Y) \in \mathcal{C}$.
Force $\otimes$-hom adjunction.

"official enrichment".

$$\mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}_{\mathcal{D}}(x,y)) = \mathrm{Hom}_{\mathcal{D}}(c \otimes x, y)$$

for $c \in \mathcal{C}^\kappa$.

Notation. $\mathrm{Rex}_\kappa = \left\{ \begin{array}{l} \text{small cats with} \\ \kappa\text{-small categories} \end{array} \right\}$.

Lurie $\otimes$. $\mathcal{D} \times \mathcal{E} \longrightarrow \mathcal{D} \otimes \mathcal{E}$ univ. property κ-cont. in each var.

$$\mathrm{Rex}_\kappa(\mathcal{C}) = \mathrm{Mod}_{\mathcal{C}^\kappa}(\mathrm{Rex}_\kappa).$$

Ex. $A \in \mathrm{CAlg}(\mathcal{C})$

$$\mathcal{C}^\kappa \times \mathrm{Mod}_A(\mathcal{C})^\kappa \qquad \mathrm{Mod}_A(\mathcal{C})^\kappa \in \mathrm{Rex}_\kappa(\mathcal{C})$$

$$M \; , \; N \longmapsto M \otimes N \qquad \text{In fact in } \mathrm{CAlg}(\mathrm{Rex}_\kappa(\mathcal{C})).$$

Get f.f. functor
$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{C}) \arrow[r, hook] & \mathrm{CAlg}(\mathrm{Rex}_\kappa(\mathcal{C})) \arrow[l, bend left, "\mathrm{End}(\mathbb{1})"] \end{tikzcd}.$$

Thm:

$$A \longmapsto \underset{\text{enriched cat}}{BA} \xmapsto{\substack{\text{formally add} \\ \kappa\text{-small colimits}}} \mathrm{Mod}_A(\mathcal{C})^{\kappa}.$$

<u>Prop</u>. $\mathrm{Rex}_\kappa(\mathcal{C})$ is again κ-presentable symmetric monoidal.

<u>Use</u>. $\mathrm{Cat} \rightleftarrows \mathrm{Rex}_\kappa \rightleftarrows \mathrm{Rex}_\kappa(\mathcal{C})$.

Follows from κ-pres. of Cat in the end.

<u>Notation</u>. $1\mathrm{Rex}_\kappa = \mathrm{Rex}_\kappa$.

$2\mathrm{Rex}_\kappa = \mathrm{Rex}_\kappa(\mathrm{Rex}_\kappa)$

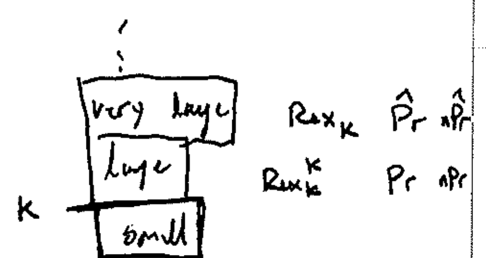
$\vdots$

$n\mathrm{Rex}_\kappa = \mathrm{Rex}_\kappa((n-1)\mathrm{Rex}_\kappa)$.

κ-cocomplete n-categories, κ-cocompleteness at every categorical level.

These are symmetric monoidal.

Now, we erase κ as follows.
From now on, we use universes.



κ-small $=$ small

$$\mathrm{Rex}_\kappa = \left\{ \begin{array}{c} \text{large cats} \\ \text{w } \kappa\text{-small} \\ \text{colimits} \end{array} \right\} =: \widehat{\Pr}$$

$$\cup \qquad\qquad \cup$$

$$\mathrm{Rex}_\kappa^\kappa = \{\text{presentable categories}\} =: \Pr$$

$\Pr \subset \widehat{\Pr}$ under $\otimes$ and small colimits

$$n\mathrm{Rex}_k = \left\{ \begin{array}{c}\text{large } n\text{-cats}\\ \text{w/ small colims}\end{array}\right\} = n\widehat{\Pr}$$

$$\cup$$

$$n\mathrm{Rex}_k^{\kappa} = \{\text{presentable } n\text{-cats}\} = n\Pr$$

Rem. $n\Pr$ is not closed under limits in general.

$$\begin{tikzcd} \mathrm{CAlg}(\widehat{\Pr}) \arrow[r, shift left] & \mathrm{CAlg}(2\widehat{\Pr}) \arrow[l, shift left] \arrow[r, shift left, "\mathcal{C} \mapsto \mathrm{Mod}_{\mathcal{C}}"] & \mathrm{CAlg}(3\widehat{\Pr}) \arrow[l, shift left, "\mathrm{End}(\mathbb{1})"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \mathrm{CatRing} \arrow[l, shift left] \end{tikzcd}$$

limit of sequence

Def. A *categorical ring* is a sequence $R = (R_1, R_2, R_3, \dots)$ where $R_n \in \mathrm{CAlg}(n\widehat{\Pr})$ and $\mathrm{End}_{R_{n+1}}(\mathbb{1}) \simeq R_n$.

Classical examples embed into CatRing. There are examples that don't come from any level.

Ex.
$$\begin{tikzcd} \mathrm{CRing} \arrow[r, hook] \arrow[d, phantom, "\wr"] & \mathrm{CAlg}(\widehat{\Pr}) \arrow[r, hook] & \mathrm{CatRing}. \\ \mathrm{Aff}^{op} \arrow[d, hook] & & \\ \mathcal{P}(\mathrm{Aff})^{op} \arrow[uurr, "\mathrm{RKE}"'] & & \end{tikzcd}$$

$$X \longmapsto \underline{\mathrm{QCoh}}(X) = (\underset{\substack{\| \\ 1\mathrm{QCoh}(X)}}{\mathrm{QCoh}(X)}, 2\mathrm{QCoh}(X), 3\mathrm{QCoh}(X), \dots)$$

Notation for target of functor.

Ex. $2\mathrm{QCoh}(X) = \mathrm{Mod}_{\mathrm{QCoh}(X)}$ if X is affine.
In general, limits of these.

Not generally the case that transitions are given by Mods.
Not seen at any finite stage.

More points for $\mathbb{G}_m$.

For qcqs schemes, determined by $1\mathrm{QCoh}(X)$.

Categorical spectra? Reason is for the "topos" reasons. Need colimits for that.