Tomer Schlank.
FRG Workshop.
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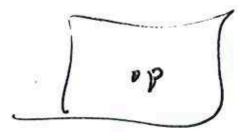
Height, additivity, Galois, and categorification.

Spec R



$$\mathrm{Sp} = \mathrm{Mod}_{\mathbb{S}}$$

$$\mathrm{Mod}_R^{k(p)\text{-ac}} \subseteq \mathrm{Mod}_R$$

$$\mathrm{Mod}_R / \mathrm{Mod}_R^{k(p)\text{-ac}} =: (\mathrm{Mod}_R)^\wedge_p .$$

p-complete.

$\underline{p=2}$.

$\vdots$	$\vdots$	$\iddots$
$(2,3)$	$(3,3)$	
$(2,2)$	$(3,2)$	
$(2,1)$	$(3,1)$	$\cdots$

0

$$\begin{array}{ccc} \mathrm{Sp} & \longrightarrow & \mathrm{Sp}_{T(h)} \\ \uplus & & \\ X & & \end{array}$$

$A \xrightarrow{f} \underline{X}$

$\sum_A f \in X$.

Joint w/ Carmeli–Yanovski and later w/ Barthel.
Also Kreidar.

A π-finite

$A \to X$

$\int_A f \in X$.

(Harpaz) $\mathcal{C} \in \mathrm{Pr}^L$

$\forall X,Y$ $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ has π-finite integration
iff $\mathcal{C}$ is a module over Pr^L over $\mathrm{Fun}^{\mathcal{S}_{\pi\text{-fin}}}(\mathrm{Corr}(\mathcal{S}_{\pi\text{-fin}}), \mathcal{S})$

$\mathcal{S}_{\pi\text{-fin}}$-shaped limits

$\mathbb{S} \longrightarrow \mathbb{S}\langle A \rangle = \mathbb{S}^A$ "cardinality measure".

$\mathbb{Z} \longrightarrow \mathbb{Z}\langle A \rangle$
$\sum_{a \in A} a$

Can use $\mathbb{S}_{T(n)}\langle A \rangle \simeq \mathbb{S}_{T(n)}^A$.

$T(n)$-local cardinality measure.

$\mathbb{S}_{T(n)}$ diagonal.

$\operatorname{colim}_A F \simeq \lim_A F$.

Harpaz in fact showed ∞-semiadditivity boils down to being a module over the mode(!) $\mathrm{Fun}^{S_{\pi\mathrm{fin}}}(\mathrm{Cat}(S_{\pi\mathrm{fin}}), S)$.

$\mathcal{C} \in \mathrm{Pr}^{L\otimes}$.

$\forall A$ π-finite. $X \in \mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(X, X)$$

$$\int_A \mathrm{id}_X \qquad X \xrightarrow{|A|} X$$

$$\underset{\substack{\| \\ p}}{|C_p|}, \ldots |BC_p|, \ldots |B^2C_p|, \ldots$$

If $|B^k C_p|$ is an iso, then $|B^{k+1} C_p|$ is its inverse.

First n s.t. $|B^n C_p|$ is an iso. $\leftarrow$ height n (or $\leq n$)

$Sp_{T(h)}$ has height h.

(Or "pure" height h.)

If $C \in \mathrm{CAlg}(\mathrm{Pr}^{L\otimes})$ has height h, (and height h addition if $\mathcal{C}$ comm.)

then $\mathrm{Mod}_C(\mathrm{Pr}^L)$ has height $h+1$. (and height $h+1$ addition if $\mathcal{C}$ comm)

What is $|A|$ in $\mathcal{C}'$?

$$\mathcal{C}' = \mathrm{Mod}_{\mathcal{C}}(P_r^L).$$

$D \in \mathcal{C}'$

$$D \xrightarrow{|A|} D$$

$$X \longmapsto \operatorname{colim}_A X.$$

So, if $|B^n C_p|$ is invertible,

then $|B^{n+1} C_p|$ is acyclic.

Like if $|G|$ is invertible,

the group homology is trivial.

$BG \qquad EG$

$$\left|\begin{matrix} G\times G\times G \\ G\times G \\ G \\ \cdot \end{matrix}\right| \qquad \left|\begin{matrix} \vdots \\ G\times G \\ G \end{matrix}\right|$$

contractible,

extra degeneracy.

If ~~????~~ $|G|$ is invertible,

can arrange choices

to get extra degeneracy

on $B.G$.

$\Pr^L_{\mathbb{C}}$ height 1.

$$\Pr^{L,\text{dual}}_{\mathbb{C}} \xrightarrow{K} Sp \qquad \text{height preservation.}$$

<u>p=2.</u> $CMon(Ani) \subseteq Sp$ ∞-semiadditivity

Assume height = 0.

$\frac{1}{2}$ CMon when 2 is invertible.

Add (-1).

$$C_2 \xrightarrow{\omega} \mathbb{1}_C^\times$$

A elementary ab. 2-group

$$\mathbb{1}[A] \xrightarrow{\sim} \mathbb{1}^{A^\vee}$$

Pontryagin - dual.

$$A \xrightarrow{x} C_2$$

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "x"'] & & \mathbb{1} \\ & C_2 \arrow[ur, "\omega"'] & \end{tikzcd}$$

Hadamard matrix.

$C \in CAlg(\Pr^{L, \infty})$ $\quad C$ ht h.

higher -1:

$$B^h C_2 \longrightarrow \mathbb{1}_C^\times \quad \text{in } Sp_{\geq 0}$$

$\mathbb{F}_2$-orientation

s.t. for every $\mathbb{F}_2$-module $M \in (Mod_{\mathbb{F}_2})_{[0,h]}$

$$\mathbb{1}_C[\Omega^\infty M] \simeq \mathbb{1}_C^{\Omega^\infty M^\vee}$$

$$Hom_{Sp}(M, \Sigma^h I_{\mathbb{Q}/\mathbb{Z}}) \cong M^\vee = Hom_{\mathbb{F}_2}(M, \mathbb{F}_2[h]).$$

"Fourier transform". Yanovski: <u>higher height semiadditivity</u>.

$\mathrm{Mod}_C^\times = \mathrm{Pic}_C$

$BC_p \longrightarrow \mathbb{1}_C^\times[1] \longrightarrow \mathrm{Pic}_C$

$$\begin{tikzcd} \mathcal{X}_h \arrow[dr] & \\ & Sp_{T(n)} \end{tikzcd}$$

universal presentable height h

Conj. $\mathcal{X}_h^{add} \simeq Sp_{T(n)}$.

h additive

infinity semiadditive of height h

$C \in \mathrm{CAlg}\left(\mathrm{Mod}_{\mathcal{X}_h^{add}}(\Pr^L)\right)$

$$\begin{tikzcd} \mathbb{1}_e^B \arrow[r] \arrow[d] \arrow[dr, phantom, "\otimes"] & \mathbb{1}_e^A \arrow[d] \\ \mathbb{1}_e^C \arrow[r] & \mathbb{1}_e^{A \times_B C} \end{tikzcd}$$

$$\begin{tikzcd} & A \arrow[d] \\ C \arrow[r] & B \end{tikzcd}$$

p fin

coCart. for $\pi_{h+1} B = 0$ $\quad \le h$

$\mathcal{S}_{p\mathrm{fin} \le h} \longrightarrow \mathcal{S}$

$A \qquad \mathrm{Hom}_{\mathrm{CAlg}}(\mathbb{1}^A, \mathbb{1})$

f lims to f lims

$\mathrm{Gal}(e) \in \mathrm{Pro}(\mathcal{S}_{p\mathrm{fin} \le h})$.

Ex. E field of char. 0

$\mathrm{Gal}(\mathrm{Mod}_{E\text{-mod}})$

$\simeq B\,\mathrm{Gal}(E)^\wedge_p$.

$$
\begin{tikzcd}
\Omega^{\mathrm{POG}} \arrow[dr] & & \mathrm{Sp}_{T(n)} \\
Z \arrow[r] & \mathbb{1}^{\times} & \mathrm{Sp}_{\geq 0}
\end{tikzcd}
$$

BG_m

GL_1

semistrict

$\mathrm{Sp}^{p\text{-tor}}_{\geq 0} \subseteq \mathrm{Sp}_{\geq 0}$ $\qquad$ $\mathbb{G}_{\mathbb{M}_p}^{ss}(\mathbb{1}^{\times})$

closed under colimits

by pfinite.

$\mathbb{G}_m^{ss}(\mathbb{1}^{\times})$ $\quad$ h-truncated

$\simeq I_{\mathbb{Q}_p/\mathbb{Z}_p}[h]$

iff $\mathrm{CAlg}(\mathcal{C})$ is h-add.