Preface to the Theory of Higher Azumaya Algebras

Lecture by Benjamin Antieau

Abstract

The Picard and Brauer spaces of a connective E_∞-ring give geometric deloopings of its space of units: they satisfy ét-descent and their connected components have concrete algebraic interpretations. The lecture reviews this picture, its twisted forms, and several calculations, then asks what survives one categorical level higher. Higher Morita categories supply formal deloopings, while cohomological classes already construct twists of the entire stack of linear stable ∞-categories. The central problem is to relate these two descriptions.

1 Units and Picard spaces

00:01:23 Throughout, R is a connective E_∞-ring spectrum. The first aim is to deloop its space of units while retaining both descent and an algebraic interpretation of connected components.

1.1 The space of units

00:01:29 **Definition 1.1.** The *space of units* of R is the homotopy pullback

$$\begin{tikzcd} \mathrm{GL}_1(R) \arrow[r] \arrow[d] & \Omega^\infty R \arrow[d, "\pi_0"] \\ (\pi_0 R)^\times \arrow[r, hook] & \pi_0 R. \end{tikzcd}$$

00:01:51 Thus $\mathrm{GL}_1(R)$ has the same positive-degree homotopy groups as R, while

$$\pi_0 \mathrm{GL}_1(R) = (\pi_0 R)^\times.$$

00:02:01 The space $\mathrm{GL}_1(R)$ is a grouplike E_∞-space, so it admits arbitrarily many abstract deloopings. The point is to find deloopings with useful geometric and algebraic properties. In particular, GL_1 is an ét-sheaf on connective E_∞-rings, whereas its pointwise classifying space is no longer a sheaf.

00:02:58 One can always sheafify such a delooping on the ét-site of connective E_∞-rings. The additional requirement is that its homotopy classes should still classify intelligible algebraic objects.

1.2 Invertible modules

00:03:18 **Definition 1.2.** The *Picard space* $\mathbf{Pic}(R)$ is the space of tensor-invertible objects in the ∞-category Mod_R of R-module spectra. Its group of components is

$$\mathrm{Pic}(R) = \pi_0 \mathbf{Pic}(R).$$

00:04:03 Pointing $\mathbf{Pic}(R)$ by the distinguished invertible object R identifies its loop space with the space of automorphisms of R:

$$\Omega_R \mathbf{Pic}(R) \simeq \mathrm{Aut}_{\mathrm{Mod}_R}(R) \simeq \mathrm{GL}_1(R). \tag{1}$$

Thus $\mathbf{Pic}(R)$ is a one-fold delooping of $\mathrm{GL}_1(R)$.

00:04:42 The components of $\mathbf{Pic}(R)$ are represented by R-line bundles, allowing also for suspensions and desuspensions.

00:05:29 Write

$$\mathrm{Aff}_{\mathbb{S}}^{\mathrm{cn}} = \left(\mathrm{CAlg}_{\mathbb{S}}^{\mathrm{cn}}\right)^{\mathrm{op}}$$

for the category of affine connective spectral schemes.

00:05:00 **Theorem 1.3** (Picard descent)**.** *The functor* $\mathbf{Pic}$ *is an* ét*-sheaf of spaces on* $\mathrm{Aff}_{\mathbb{S}}^{\mathrm{cn}}$*. There is a fiber sequence of* ét*-sheaves*

$$B\mathrm{GL}_1 \longrightarrow \mathbf{Pic} \longrightarrow \underline{\mathbb{Z}}. \tag{2}$$

For every connective E_∞-ring R,

$$\mathrm{Pic}(R) \cong H^0_{\text{ét}}(\operatorname{Spec} \pi_0 R, \mathbb{Z}) \times H^1_{\text{ét}}(\operatorname{Spec} \pi_0 R, \mathbb{G}_m). \tag{3}$$

00:06:11 The second factor in equation (3) is the group of line bundles on the classical scheme $\operatorname{Spec} \pi_0 R$, while the first records their locally constant shifts. For the sphere, this description goes back to Hopkins–Mahowald–Sadofsky [HMS94].

00:06:45 The Picard space therefore has both desired properties: it is an ét-sheaf, and its connected components have concrete representatives in Mod_R.

2 Azumaya algebras and the Brauer space

00:07:22 The next problem is to deloop once more. The pointwise classifying space $B\mathbf{Pic}(R)$ is again not *a priori* a sheaf, so a geometric construction requires more care.

2.1 Azumaya algebras and Morita invertibility

00:08:16 **Definition 2.1.** An E_1-R-algebra A is *Azumaya* if

(i) A is a compact generator of Mod_R; and

(ii) the canonical map

$$A^{\mathrm{op}} \otimes_R A \longrightarrow \mathrm{End}_R(A) \tag{4}$$

is an equivalence.

00:07:55 For ring spectra, this definition is due to Baker–Richter–Szymik [BRS12]; it transports earlier definitions from dg algebras and classical algebra, going back through Toën, Grothendieck, Auslander–Goldman, and Azumaya.

00:09:00 Because A compactly generates Mod_R, it is derived Morita equivalent to R. In the relevant Morita category, A^{op} acts as an inverse to A.

00:09:16 **Definition 2.2.** Let $\mathbf{Cat}_R$ be the symmetric monoidal ∞-category of small, stable, idempotent-complete R-linear ∞-categories. Its tensor unit is $\mathrm{Perf}(R)$. The *Brauer space* of R is

$$\mathbf{Br}(R) = \mathbf{Pic}(\mathbf{Cat}_R),$$

and its *Brauer group* is

$$\mathrm{Br}(R) = \pi_0\mathbf{Br}(R).$$

00:09:58 For an Azumaya algebra A, Morita invertibility takes the form

$$\mathrm{Perf}(A^{\mathrm{op}}) \otimes_R \mathrm{Perf}(A) \simeq \mathrm{Perf}(R). \tag{5}$$

Antieau briefly hesitated over the order of the two factors on the board; the point was that the two perfect-module categories are tensor inverses.

00:10:22 The automorphisms of the unit $\mathrm{Perf}(R)$ are its R-linear autoequivalences. Such a functor is determined by the image of R, which must be an invertible R-module. Consequently,

$$\Omega_{\mathrm{Perf}(R)}\mathbf{Br}(R) \simeq \mathbf{Pic}(R). \tag{6}$$

The Brauer space is thus a twofold delooping of $\mathrm{GL}_1(R)$.

00:11:06 Its components again have algebraic representatives: Azumaya algebras, or their categories of perfect modules. Over a field, central simple algebras are the basic examples.

2.2 Descent and the cohomological calculation

00:11:27 **Theorem 2.3** (Toën; Antieau–Gepner)**.** *On* $\mathrm{Aff}^{\mathrm{cn}}_{\mathbb{S}}$*:*

(i) $\mathbf{Br}$ *is an* ét*-sheaf of spaces;*

(ii) *every Azumaya algebra is* ét*-locally trivial in the Morita sense.*

For simplicial commutative rings this is due to Toën, and for connective E_∞-rings to Antieau–Gepner [Toe11; AG12].

00:12:12 Here "trivial" does not mean isomorphic to R as an algebra: it means Morita equivalent to the base. Classically, the corresponding assertion is that an Azumaya algebra is locally a matrix algebra.

00:12:32 The two parts of Theorem 2.3 require largely separate arguments. The sheaf statement uses higher stacks, deformation theory, and derived algebraic geometry; the local-triviality argument is more closely related to the behavior of triangulated categories.

00:13:11 Since every component of $\mathbf{Br}$ is locally the neutral component,

$$\mathbf{Br} \simeq B\mathbf{Pic} \qquad \text{in ét-sheaves.} \tag{7}$$

This is a sheafwise statement, not a claim that $\mathbf{Br}(R) \simeq B\mathbf{Pic}(R)$ pointwise.

00:14:04 **Corollary 2.4.** *For every connective E_∞-ring R,*

$$\mathrm{Br}(R) \cong H^1_{\text{ét}}(\operatorname{Spec}\pi_0 R, \mathbb{Z}) \times H^2_{\text{ét}}(\operatorname{Spec}\pi_0 R, \mathbb{G}_m). \tag{8}$$

00:14:37 Relative to equation (3), both cohomological degrees have shifted upward by one. The Brauer space is a sheaf, and its global components are represented by Azumaya algebras. The goal is to continue this pattern to still higher deloopings.

3 Twisted categories and Brauer-group calculations

3.1 Twisted sheaves and twisted K-theory

00:16:53 The Brauer group $\mathrm{Br}(R) = \pi_0\mathbf{Br}(R)$ is a group of twists. Its elements twist both module categories and invariants of those categories.

00:17:12 If A is an Azumaya R-algebra, then Mod_A is an ét-twisted form of Mod_R: there is an ét-cover $R \to S$ such that

$$\mathrm{Mod}_A \otimes_R S \simeq \mathrm{Mod}_S \,. \tag{9}$$

00:17:55 **Definition 3.1.** Given a cohomological Brauer class

$$\alpha \in H^1_{\text{ét}}(\operatorname{Spec} \pi_0 R, \mathbb{Z}) \times H^2_{\text{ét}}(\operatorname{Spec} \pi_0 R, \mathbb{G}_m), \tag{10}$$

let Mod_R^α denote the stable ∞-category of *α-twisted sheaves* over R.

00:18:18 The category Mod_R^α is locally equivalent to Mod_R, but the local pieces are glued using the transition data encoded by α.

00:18:43 The key step in proving representability by an Azumaya algebra is to show that Mod_R^α has a single compact generator. If E is such a generator, then

$$\mathrm{End}_{\mathrm{Mod}_R^\alpha}(E) \tag{11}$$

is an Azumaya algebra representing α. The argument successively glues local generators to obtain a global one; this input is needed in the descent theorem above.

00:19:25 **Definition 3.2.** The *α-twisted algebraic K-theory* of R is

$$\mathrm{K}^\alpha(R) = \mathrm{K}\big((\mathrm{Mod}_R^\alpha)^\omega\big). \tag{12}$$

If α is represented by an Azumaya algebra A, this is the algebraic K-theory of the compact objects in Mod_A, namely $\mathrm{K}(\mathrm{Perf}(A))$.

00:20:05 Thus the classes in equation (10) twist algebraic K-theory, just as classes in singular $H^3(-, \mathbb{Z})$ twist topological K-theory. Antieau emphasized that there is a precise connection between the two constructions.

00:20:27 *Remark* 3.3. If the twist lies entirely in the $H^2_{\text{ét}}(-, \mathbb{G}_m)$ factor, then the canonical t-structure on Mod_R descends to Mod_R^α: no shifting automorphisms from the $\mathbb{Z}$-factor enter the gluing.

3.2 Connective arithmetic examples

00:20:48 **Theorem 3.4** (Antieau–Gepner)**.** *The sphere and its basic arithmetic localizations have Brauer groups*

$$\mathrm{Br}(\mathbb{S}) = 0, \tag{13}$$
$$\mathrm{Br}\big(\mathbb{S}[1/p]\big) \cong \mathbb{Z}/2, \tag{14}$$

and the Brauer group of the p-local sphere fits into the exact sequence

$$0 \longrightarrow \mathrm{Br}(\mathbb{S}_{(p)}) \longrightarrow \mathbb{Z}/2 \oplus \bigoplus_{\ell \neq p} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0. \tag{15}$$

00:20:56 The vanishing of $\mathrm{Br}(\mathbb{S})$ was conjectured by Baker–Richter. Once Theorem 2.3 is known, these assertions follow from the corresponding ét-cohomology calculations and class field theory [AG12].

00:21:18 The identity equation (13) says that an $\mathbb{S}$-linear stable ∞-category which is ét-locally equivalent to spectra is already globally equivalent to spectra. In this sense, the theory of spectra has no nontrivial ét-twisted forms.

00:22:28 By contrast, $\mathrm{Br}(\mathbb{S}_{(p)})$ is large: its primary torsion is infinite at every prime. Hence there are stable ∞-categories over $\mathbb{S}_{(p)}$ which are ét-locally p-local spectra but are not globally equivalent to p-local spectra.

00:22:42 Experience from algebraic geometry suggests that nonzero Brauer classes should appear as obstructions in moduli problems. The challenge is to find constructions in which an expected spectrum is replaced naturally by a twisted spectrum.

00:23:43 **Proposition 3.5** (Problem)**.** *Fix $\alpha \in \mathrm{Br}(\mathbb{S}_{(p)})$. Construct an unstable ∞-category $\mathcal{S}^{\alpha}_{(p)}$ of* α-twisted p-local spaces*, with p-locality understood in the sense of Bousfield localization, such that its category of cohomology theories is*

$$\mathrm{CohTh}(\mathcal{S}^{\alpha}_{(p)}) \simeq \mathrm{Mod}^{\alpha}_{\mathbb{S}_{(p)}}\,. \tag{16}$$

00:25:00 The thick subcategories of compact objects in $\mathrm{Mod}^{\alpha}_{\mathbb{S}_{(p)}}$ can be classified by chromatic height, along the same lines as the Devinatz–Hopkins–Smith classification. Thus the twisted category behaves much like p-local spectra without being p-local spectra.

00:25:49 *Remark* 3.6. Asked about a direct relation between the corresponding unstable p-local spaces and p-local spectra, Antieau answered that he did not know one. He noted, however, that the twists under discussion arise from much smaller groups of automorphisms and are torsion; without the torsion condition he would have regarded the proposal as implausible.

3.3 Nonconnective spectra and Galois descent

00:26:54 There is substantial work on Picard groups, and increasingly on Brauer groups, of nonconnective ring spectra. These questions are harder because the ét-topology is less effective in the nonconnective setting: it fails to detect phenomena that are visible by other forms of descent.

00:27:42 **Example 3.7** (Gepner–Lawson)**.** There is a nonzero class

$$0 \neq \beta \in \mathrm{Br}(KO)$$

which is split by the C_2-Galois extension $KO \to KU$ in the sense of Rognes. That is, after base change to KU the associated Azumaya algebra is Morita trivial. The map $KO \to KU$ is not an ét-cover [GL16].

00:28:24 *Remark* 3.8. Antieau proposed that $\mathrm{Br}(KU) = 0$, saying that he knew no counterexample and that many people expected the assertion to hold. An audience member said that this was known for the ordinary smash product. No proof or reference was given, and Antieau left its status as “conjecture, possibly theorem.”

00:29:06 This suggests seeking a Galois–ét topology, combining ét-covers with Galois covers, in which Brauer classes become locally trivial for a large class of suitably geometric ring spectra. Spectra built from TMF were the guiding examples; Antieau did not expect such a result for arbitrary ring spectra.

4 The threefold-delooping problem

00:29:50 To expose what is missing at the next level, Antieau organized the known and conjectural interpretations of $B\mathrm{GL}_1$, $B^2\mathrm{GL}_1$, and $B^3\mathrm{GL}_1$ as in Table 1.

	Algebraic	Categorical	Stack-theoretic	Cohomological	Twisted invariant
$B\mathrm{GL}_1$	line bundles	$\mathbf{Pic}(\mathrm{Mod}_R)$	twists of $\mathcal{O}_X^\times$; $\mathbb{G}_m$-torsors	$H^0_{\text{ét}}(\mathbb{Z}) \times H^1_{\text{ét}}(\mathbb{G}_m)$	$H^*(X, \mathcal{O}_X)$
$B^2\mathrm{GL}_1$	Azumaya algebras	$\mathbf{Pic}(\mathbf{Cat}_R)$	twists of Mod_R	$H^1_{\text{ét}}(\mathbb{Z}) \times H^2_{\text{ét}}(\mathbb{G}_m)$	algebraic K-theory
$B^3\mathrm{GL}_1$	unknown	unknown	twists of $\mathbf{Cat}_R$	$H^2_{\text{ét}}(\mathbb{Z}) \times H^3_{\text{ét}}(\mathbb{G}_m)$	$K^{(2)}(R)$; perhaps TMF???

Table 1: Known and proposed interpretations of the first three deloopings. The cohomology is taken on $\operatorname{Spec} \pi_0 R$.

00:30:54 At the first level, line bundles form the Picard space of R-modules and are the same stack-theoretic data as $\mathbb{G}_m$-torsors; a line bundle twists the cohomology of $\mathcal{O}_X$. At the second, Azumaya algebras form the Picard space of R-linear categories, twist Mod_R, and thereby twist algebraic K-theory.

00:32:32 At the third level, the algebraic and categorical entries are the fundamental unknowns, and they are closely related. The stack-theoretic and cohomological entries appear more accessible: one expects twists of the stack $\mathbf{Cat}_R$ governed by

$$H^2_{\text{ét}}(\operatorname{Spec} \pi_0 R, \mathbb{Z}) \times H^3_{\text{ét}}(\operatorname{Spec} \pi_0 R, \mathbb{G}_m).$$

00:33:30 The natural invariant to twist is some form of secondary K-theory. Antieau was deliberately agnostic among the several available definitions, expecting any of them to admit such twists.

00:33:48 *Remark* 4.1. There is a more speculative chromatic analogy: ordinary cohomology behaves like a height-zero theory and algebraic K-theory like a height-one theory, so perhaps the third-level classes act on something related to TMF. Antieau described this possibility as "pretty out there."

00:34:12 The unknown algebraic entry is a version of the same problem raised earlier in the conference in terms of fully dualizable 3-categorical objects. The strategy for the remainder of the lecture is to set it aside temporarily and study the stack-theoretic twists as though an algebraic interpretation were already available.

4.1 Higher Morita theory of E_n-algebras

00:35:07 There is one optimistically definitive construction in the literature, but present knowledge does not show that it has all the desired geometric properties. Indeed, Antieau stressed that almost nothing is currently known about its connected components.

00:35:35 The construction is Haugseng's theory of higher Morita categories [Hau20].

00:35:48 **Definition 4.2.** For an E_∞-ring R, let

$$\mathrm{Alg}_n^{\mathrm{Mor}}(R) \tag{17}$$

denote Haugseng's $(\infty, n+1)$-category whose objects are E_n-R-algebras. The 1-morphisms from A to B are E_{n-1}-algebras in the category of (A, B)-bimodules; the next morphisms are E_{n-2}-algebras in the appropriate bimodule category, and so on.

00:36:58 For $n = 2$, this construction recovers the part of $\mathbf{Cat}_R$ consisting of stable R-linear ∞-categories generated by a single object.

00:37:37 **Definition 4.3.** The *nth Brauer space* is

$$\mathbf{Br}_n(R) = \mathbf{Pic}\big(\mathrm{Alg}_n^{\mathrm{Mor}}(R)\big). \tag{18}$$

Thus one retains the tensor-invertible objects and all invertible higher morphisms. Its group of components

$$\mathrm{Br}_n(R) = \pi_0 \mathbf{Br}_n(R)$$

is the *n-ary Brauer group* of R.

00:38:06 **Theorem 4.4** (Haugseng)**.** *With* $\mathbf{Br}_0(R) = \mathbf{Pic}(R)$*, there are equivalences*

$$\Omega_R \mathbf{Br}_n(R) \simeq \mathbf{Br}_{n-1}(R). \tag{19}$$

00:38:22 Haugseng's construction therefore produces deloopings of $\mathrm{GL}_1(R)$ in all degrees. This is one of the two properties sought: the remaining issues are whether the spaces satisfy descent and whether their components have useful algebraic representatives.

4.2 Four open questions

00:39:15 **Proposition 4.5** (Question I)**.** *For some* $n \geq 2$ *and some* E_∞*-ring* R*, is*

$$\mathrm{Br}_n(R) \neq 0?$$

Antieau said that no such example was known.

00:39:52 Cohomology would supply candidate classes if one already knew that $\mathbf{Br}_n$ were a further delooping of the ordinary Brauer sheaf. The problem is that Definition 4.3 is purely algebraic: one must exhibit an invertible E_n-algebra in the higher Morita category.

00:40:43 **Proposition 4.6** (Question II)**.** *Are there "trivial" n-ary Azumaya algebras – E_n-algebras not equivalent to R as algebras but equivalent to R in the higher Morita category? Already for $n = 2$, no analogue of the plentiful matrix-algebra examples was clear.*

00:41:34 **Proposition 4.7** (Question III)**.** *Is every n-ary Azumaya algebra* ét*-locally trivial? Such a result would be needed to compare* $\mathbf{Br}_n$ *with the delooping of* $\mathbf{Br}_{n-1}$ *in* ét*-sheaves.*

00:41:59 **Proposition 4.8** (Question IV)**.** *Does* $\mathbf{Br}_n$ *satisfy general* ét*-descent? Together with Proposition 4.7, this would yield the desired sheafwise equivalence with a delooping of* $\mathbf{Br}_{n-1}$*.*

00:42:09 *Remark* 4.9. Antieau thought it might be possible to prove that $\mathbf{Br}_n$ is a separated prestack, in the sense that it behaves correctly away from degree zero. The problem of components would remain.

00:42:29 *Remark* 4.10. Asked for evidence for ét-local triviality, Antieau said that he had none beyond analogy with line bundles and ordinary Azumaya algebras. Even in the ordinary Brauer case, local triviality is not obvious.

00:43:24 Antieau's personal suspicion was that higher Morita categories of E_n-algebras are not the correct framework for these higher twists. He therefore turned to what can be constructed directly from cohomology.

5 Cohomological higher twists of linear categories

00:43:42 **Definition 5.1.** The *secondary cohomological Brauer group* of a connective E_∞-ring R is

$$\mathrm{Br}_2^{\mathrm{coh}}(R) = H^2_{\text{ét}}(\operatorname{Spec} \pi_0 R, \mathbb{Z}) \times H^3_{\text{ét}}(\operatorname{Spec} \pi_0 R, \mathbb{G}_m). \tag{20}$$

00:44:05 **Theorem 5.2** (Cohomological twisting)**.** *For every $\sigma \in \mathrm{Br}_2^{\mathrm{coh}}(R)$, there is a twisted form $\mathbf{Cat}^\sigma$ of the stack* **Cat** *of linear stable ∞-categories. The objects of its global sections*

$$\mathbf{Cat}^\sigma(R)$$

are called σ-twisted stable ∞-categories.

00:45:19 More explicitly, **Cat** assigns to an R-algebra S the category of S-linear stable ∞-categories. An object of $\mathbf{Cat}^\sigma(R)$ is ét-locally an ordinary stable ∞-category, but its local pieces are glued by the transition data of σ, just as α-twisted sheaves are locally ordinary sheaves with twisted gluing.

00:45:53 It makes sense to consider exact functors between σ-twisted categories. Conditions such as smoothness, properness, and homotopical finite presentation are imposed locally. Consequently, constructions available for ordinary stable ∞-categories can be repeated in the twisted setting.

5.1 Twisted motives and secondary K-theory

00:46:53 Passing from a cohomological Brauer class to an explicit Azumaya algebra is typically difficult. Antieau emphasized that this difficulty already encodes deep questions in arithmetic geometry and several classical results of class field theory. Although the cohomological and Azumaya descriptions give equivalent groups, only one direction is readily explicit, and a single class has many Morita-equivalent Azumaya representatives.

00:47:57 From $\mathbf{Cat}^\sigma$ one can construct a category of σ-twisted noncommutative motives and a chosen version of σ-twisted secondary K-theory, following the corresponding untwisted constructions.

00:48:22 The striking difference from the ordinary Brauer group is that

$$\mathrm{Br}(\mathbb{S}) = 0, \qquad \mathrm{Br}_2^{\mathrm{coh}}(\mathbb{S}) \neq 0.$$

Thus the theory of spectra has no nontrivial ordinary ét-twisted form, but the entire theory of stable ∞-categories does admit nontrivial cohomological twists over the sphere.

5.2 A Mayer–Vietoris twist over the sphere

00:48:53 **Example 5.3.** Fix distinct primes p and q. The affine opens associated with inverting p and q give a cover of $\operatorname{Spec} \mathbb{S}$ with intersection $\operatorname{Spec} \mathbb{S}[1/pq]$:

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{S}[1/pq] \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{S}[1/q] \arrow[d] \\
\operatorname{Spec} \mathbb{S}[1/p] \arrow[r] & \operatorname{Spec} \mathbb{S}.
\end{tikzcd}$$

The corresponding part of the Mayer–Vietoris sequence is

$$0 = \mathrm{Br}(\mathbb{S}) \longrightarrow \mathrm{Br}\big(\mathbb{S}[1/p]\big) \oplus \mathrm{Br}\big(\mathbb{S}[1/q]\big) \longrightarrow \mathrm{Br}\big(\mathbb{S}[1/pq]\big) \xrightarrow{\delta} H^3_{\text{ét}}(\operatorname{Spec} \mathbb{S}, \mathbb{G}_m). \tag{21}$$

The first direct sum is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, whereas $\mathrm{Br}(\mathbb{S}[1/pq])$ is infinite and contains a $\mathbb{Q}/\mathbb{Z}$ summand. Hence the boundary map δ is nonzero. Choose

$$\alpha \in \mathrm{Br}\big(\mathbb{S}[1/pq]\big), \qquad \delta\alpha \neq 0,$$

and suppose that α is represented by an Azumaya $\mathbb{S}[1/pq]$-algebra A.

An object of $\mathbf{Cat}^{\delta\alpha}(\mathbb{S})$ is a triple $(\mathcal{M}, \mathcal{N}, \varphi)$ consisting of an $\mathbb{S}[1/p]$-linear stable ∞-category $\mathcal{M}$, an $\mathbb{S}[1/q]$-linear stable ∞-category $\mathcal{N}$, and an equivalence

$$\varphi\colon \mathcal{M}[1/q] \otimes_{\mathbb{S}[1/pq]} \mathrm{Mod}_A \xrightarrow{\simeq} \mathcal{N}[1/p]. \tag{22}$$

00:50:58 Ordinary descent would glue $\mathcal{M}[1/q]$ directly to $\mathcal{N}[1/p]$. In equation (22), the identification is modified by tensoring with the invertible $\mathbb{S}[1/pq]$-linear category Mod_A. This gives a twisted integral form of the entire theory of stable ∞-categories.

5.3 Representability and finiteness

00:51:39 **Proposition 5.4** (Representability problem)**.** *For the class $\delta\alpha$ above, is there an E_2-algebra B such that*

$$\mathbf{Cat}_{\mathbb{S}}^{\delta\alpha} \simeq \mathbf{Cat}_B? \tag{23}$$

More flexibly, can $\mathbf{Cat}_{\mathbb{S}}^{\delta\alpha}$ be represented as $\mathbf{Cat}_{\mathcal{C}}$ for some stable ∞-category $\mathcal{C}$? Antieau thought the second formulation had a somewhat better chance of succeeding.

00:52:15 **Proposition 5.5** (Finiteness problem)**.** *Does $\mathbf{Cat}^{\delta\alpha}(\mathbb{S})$ contain a smooth and proper category? Antieau did not know the answer. He did know that it contains homotopically finitely presented categories, so the twisted theory is not vacuous, although it could still be extremely small.*

00:52:54 The cohomological construction is concrete enough to work with directly, even before an algebraic representative is known. Antieau regarded this as more promising than beginning with an unknown kind of higher Azumaya algebra.

00:53:26 *Remark* 5.6. In response to a final question, Antieau clarified that the E_2 case is exactly the threefold-delooping problem for $\mathrm{GL}_1(\mathbb{S})$. The class $\delta\alpha$ lies in π_0 of a particular threefold delooping of the unit spectrum; the unresolved task is to understand its algebraic meaning.

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