Pertusi.

3.

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Moduli interpretation of some HKs.

Thm (Li-P—Zhao). $F(Y) \simeq M_\sigma(\mathrm{Ku}\,Y, \lambda_1+\lambda_2)$.

If $Y \not\ni$ plane, $Z(Y) \simeq M_\sigma(\mathrm{Ku}\,Y, 2\lambda_1+\lambda_2)$.

LLSvS 8fold

Idea of pf (of (2)).

1) Objects. $C \subset Y$ twisted cubic

linear span $\langle C \rangle \cong \mathbb{P}^2$

intersect with Y gives a cubic surface.

$$\begin{array}{ccc} M_3(Y) & \xrightarrow{\mathbb{P}^2\text{ bundle}} & M'(Y) \quad 8\text{ dim} \\ & & \downarrow \text{ contraction of divisor} \\ & & Z(Y) \end{array}$$

twisted cubic curves $\quad$ 10 dim

$I_{C/S}(2)$ $\qquad D^b(Y) \simeq \langle \mathrm{Ku}\,Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$

$\simeq_{\text{mutate}} \langle \mathcal{O}(-1), \mathrm{Ku}\,Y, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle$

$\mathbb{L}_{\mathcal{O}_Y(1)}$ left mutation

$F_C := \mathbb{L}_{\mathcal{O}_Y}(I_{C/S}(2))$

If C is nice, then $F_C \in \mathrm{Ku}\,Y$.

In general, $\mathbb{R}_{\mathcal{O}(-1)} F_C \in \mathrm{Ku}\,Y$.

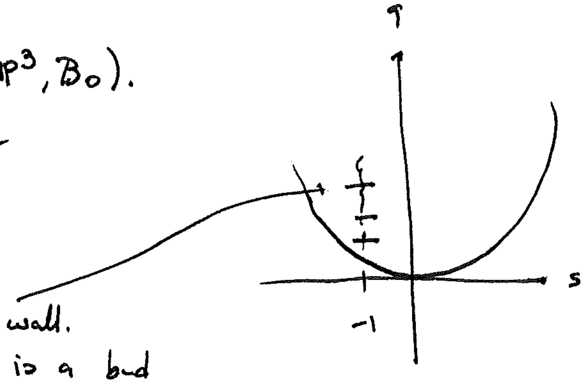
Then has numerical class $2\lambda_1+\lambda_2$. Need to check stability.

Go to $D^b(\mathbb{P}^3, \mathcal{B}_0) \rightsquigarrow \sigma_{-1,9}$ (OK even though σ is a small rotation.)

E_C = essential image of F_C in $D^b(\mathbb{P}^3, \mathcal{B}_0)$.

E_C' = ... F_C' ...

E_C is σ-stable for $t \gg 0$.



Compute walls.

First wall.
If C is a bad curve, then E_C is unstable and E_C' becomes stable.

Contraction of $M'(Y) \to Z(Y)$ "is" a wall crossing.

Two more walls, but they are not relevant.

$F(Y)$ is similar.

OG10 case. $v_0 \in K_0^{num}(Ku\,Y)$ = algebraic Mukai lattice.

primitive, $v_0^2 = 2$, $v = 2v_0$ $(v^2 = 8)$.

$\sigma \in \mathrm{Stab}^+(Ku\,Y)$ v-generic: strictly semistable locus is $\mathrm{Sym}^2(M_\sigma(v_0))$

ex $I_Z \oplus I_{Z'}$

(LPZ)

<u>Thm</u>. $M_\sigma(v)$ has a symplectic resolution $\tilde{M}$ obtained by blowing up sing. locus (strictly semistable locus) w/ reduced scheme structure and $\tilde{M}$ is a HK 10-fold $\underset{\text{def}}{\sim}$ OG10.

Similar to Lehn–Sorger.

<u>Application</u>. $v_0 = \lambda_1 + \lambda_2$, $v = 2v_0$, so $\tilde{M} \longrightarrow M_\sigma(v)$ symplectic resolution.

Why take this v_0? Cubic 3Folds and intermediate Jacobians.

[D–, Beauville, MT] X a cubic 3fold.

$$\begin{array}{ll} & M_{inst,X} = \text{moduli space of instanton sheaves } F \text{ on } X \\ & \quad\quad \text{i.e. semistable sheaves } F \text{ with } (2,0,-2,0) = ch(F). \\ & \Big\downarrow c_2 \\ \text{Intermediate Jacobian} & J^1(X) = \text{1-cycles of deg 1 on } X \end{array}$$

1) Description of sheaves in $M_{inst,X}$.

- $\Gamma \subset X$ elliptic quintic curve in X, deg 5, $\omega_\Gamma = \mathcal{O}_\Gamma$, $h^0(\mathcal{O}_\Gamma) = 1$, $\langle \Gamma \rangle = \mathbb{P}^4$.

$$0 \to \mathcal{O}_X(-1) \to F_\Gamma \to I_{\Gamma/X}(1) \to 0$$

$\uparrow$ unique extension and F_Γ is slope <u>stable</u>, F_Γ is rank 2.

- $C \subset X$ a smooth conic. Similar const.
  gives F_C <u>stable</u>, rank 2 torsion free.

- Semistables are $I_{l_1/X} \oplus I_{l_2/X}$, l_1, l_2 lines in X.
  (strictly semistables)

<u>Thm</u> (DBMT). $M_{inst,X} = Bl_{F^2} J^1(X)$

$\uparrow$ image of conic locus

Back to cubic 4Folds.

$F \in M_{inst,X} \rightsquigarrow ch(i_* F) = 2\lambda_1 + 2\lambda_2 \; (= v)$.

$X \overset{i}{\hookrightarrow} Y$ cubic 4fold.

cubic 3fold

Want to relate to $Ku\,Y$.

<u>Objects</u>. Γ elliptic quintic in Y C conic

$pr = \mathbb{R}_{\mathcal{O}_Y(-1)} \circ \mathbb{R}_{\mathcal{O}_Y(-2)} \circ \mathbb{L}_{\mathcal{O}_Y}$

$E_\Gamma := pr(I_{\Gamma/Y}(1))$.

$E_C := pr(i_* F_C)$.

<u>Thm</u>. E_Γ and E_C are stable in $Ku\,Y$.

$\underset{i_* F_\Gamma}{\overset{E_\Gamma}{\parallel}}$,

$i : X \hookrightarrow Y$

where $X = Y \cap \langle \Gamma \rangle$.
$\qquad\qquad\qquad \underset{\mathbb{P}^4}{\cap}$

Strict semistable locus $\cong Sym^2 F(Y)$.

$M_\sigma(\lambda_1 + \lambda_2) \cong F(Y)$,

<u>Intermediate Jacobians</u>. Family $\mathbb{P}_0$ of smooth hyperplane sections of Y.

$\rightsquigarrow$ Family of intermediate Jacobians.

$$\begin{tikzcd} & \mathcal{X} \arrow[d] & \mathcal{J}(Y) \arrow[dl] \\ (\mathbb{P}^5)^\vee \supseteq & \mathbb{P}_0 & \end{tikzcd}$$

with fibers J of $\mathcal{X}_t$.

[Donagi–Markman] $\mathcal{J}(Y)$ has a symplectic form. $\underline{\underline{Q}}$ is there a HK compactification $\overline{\mathcal{J}(Y)}$ plus a Lagrangian fibration over $(\mathbb{P}^5)^\vee$.

<u>Thm</u> [Laza–Saccà–Voisin]. Proved for Y very general and untwisted family.

[Voisin] Y very general and twisted family.

[Saccà] All (smooth) Y using MMP!

Our case: $M_\sigma(2\lambda_1+2\lambda_2) \leftarrow \tilde{M}$

$\tilde{M} \supset M_0 \cong \{E_\Gamma,\ \Gamma \text{ elliptic quintic in } Y\}$ open

$\tilde{M} \underset{\text{rational}}{\rightsquigarrow} \mathbb{P}_0$

$E_\Gamma \longmapsto \langle \Gamma \rangle \cap Y = X$

Compactify to Lagrangian fibration.

<u>Thm</u> $\exists$ proj. HK N birational to $\tilde{M}$ together w/ Lagrangian fibration to B compactifying $\mathcal{J}(X) \rightsquigarrow \mathbb{P}_0$.

<u>Work in progress</u>. Something about 2nd gen. Kummers.

Joint w/ Bayer, Perry, Zhao.

$\nexists$ explicit description of general def. of

GK^n type

Goal: (1) Construct a 4-dim family of nc abelian surfaces.

(2) Const σ.

(3) Moduli spaces.