Pertusi.

2.

21 October 2023.

$Y \subseteq \mathbb{P}^5$ cubic fourfold.

[Kuznetsov] semiorthogonal decomposition.

$$D^b(Y) = \langle \mathrm{Ku}\,Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$$

3 line bundles, exceptional collection

$\mathrm{Hom}(\mathcal{O}_Y(i), \mathcal{O}_Y(j)[k]) = 0 \quad i > j$ or $i = j$ $k \neq 0$ and $\mathbb{C}$ $i=j, k=0$.

Kuznetsov component;

right orthogonal $= \{F \in D^b(Y) : \mathrm{Hom}(\mathcal{O}_Y(i), F[k]) = 0 \quad \forall i = 0,1,2, \forall k \in \mathbb{Z}\}$.

$$G \in D^b(Y) \Rightarrow \quad \underset{\langle \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)\rangle}{\underset{\cap}{G_2}} \to G \to \underset{\mathrm{Ku}\,Y}{\underset{\cap}{G_1}} \to G_2[1]$$

Using right adjoints.

Ex. $\langle \mathcal{O}_Y(2) \rangle \xrightarrow{\alpha} D^b(Y)$, $\alpha^!$ right adjoint

$$\alpha^!(G) = \bigoplus \mathrm{Hom}(\mathcal{O}_Y(2), G[k]) \otimes \mathcal{O}_Y(2)[-k]$$

$\vdots$

left mutated...

Properties. ① Serre functor $S_{\mathrm{Ku}Y} = [2]$, so $\mathrm{Ku}Y$ is a 2-CY category.

$$S_{\mathrm{Ku}Y}^{-1} = \mathbb{L}_{\langle \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)\rangle}\left(- \otimes \mathcal{O}_Y(3)\right)[-4] \underset{\text{check}}{\simeq} [-2].$$

② $\mathrm{Ku}Y$ has the same HH as D^b(K3 surface). (what about in char. 2, 3?)

③ There are rational cubic fourfolds such that $\mathrm{Ku}Y \simeq D^b(S)$, S a K3.

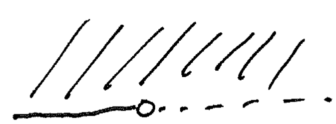
Few words about stability conditions. $\mathcal{Z} \underset{\text{admissible}}{\hookrightarrow} D^b(X)$, X a sm. proj.

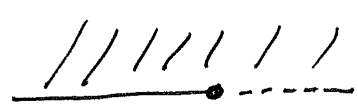
$K(\mathcal{Z})$ = Grothendieck group of $\mathcal{Z}$.

A (weak) stability condition on $\mathcal{Z}$ is $\sigma = (\mathcal{A}, Z)$ where

- $\mathcal{A}$ is the heart of a bounded t-structure,
- Z a (weak) stability function

$$Z : \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ K(\mathcal{A})}}{K(\mathcal{Z})} \longrightarrow \mathbb{C}$$

where $Z(K(\mathcal{Z}) \setminus 0) =$ 

or 

(weak case).

So, for weak, some can map to zero.

HN property filt-tn by stables

+ support property

Slope : $E \in \mathcal{A}$, $\mu_\sigma(E) = \begin{cases} -\dfrac{Re(z)}{Im(z)} & \text{if } Im(z) > 0, \\ +\infty & \text{if } Im(z) = 0. \end{cases}$

E is <u>σ-semistable</u> if $\forall F \hookrightarrow E \Rightarrow \mu_\sigma(F) \leq \mu_\sigma(E)$

<u>σ-stable</u> if $\mu_\sigma(F) < \mu_\sigma(E/F)$ — or $\mu_\sigma(E)$ for non-weak stab. conditions

$F \in \mathcal{Z}$ is σ-semistable, etc

if some shift is in the heart and is such.

<u>Exs</u>.

① Slope stability, $D^b(X)$, X sm. proj., H an ample class, $n = \dim X$

$$\sigma_{slope} = \left(Coh(X), Z_{slope} = -\left(ch_1(-)H^{n-1}\right) + \sqrt{-1}\, rank(-) H^n \right).$$

Claim: this is a weak stability condition, which is not weak if $n = 1$.

② Tilt stability. $s, q \in \mathbb{R}$, $q > \frac{1}{2}s^2$

$$T^s = \langle F \in Coh X,\ F \text{ is } \sigma_{slope}\text{-semistable},\ \mu_{slope}(F) > s \rangle$$

(T^s does not map to F^s.)

$$F^s = \langle \qquad \text{————————} \qquad \leq s \rangle$$

$Coh^s(X) = \langle T^s, F^s[1] \rangle$ is the heart of a bounded t-structure.

$$\mathcal{H}^{-1}(F)[1] \to F \to \mathcal{H}^0(F), \quad F \in Coh^s(X).$$

$$Z_{s,q}(F) = -ch_2(F)H^{n-2} - q\, rank(F) H^n + \sqrt{-1}\left(ch_1(F)H^{n-1} - s\, rank(F) H^n\right).$$

$q > \frac{1}{2}s^2$ relates to Bogomolov inequality.

<u>Prop</u>. (BMT, BMS). $\sigma_{s,q}$ is weak stable, non-weak stable if $n = 2$.

Criterion. $D^b(X) = \langle \mathcal{Z}, E_1, \dots, E_m \rangle$

— E_i exceptional

— $\sigma = (\mathcal{A}, Z)$ a weak stab. cond. on $D^b(X)$.

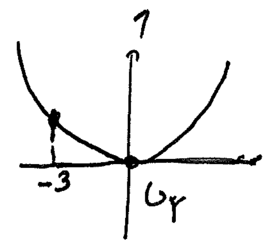
Assume:

- $E_i \in \mathcal{A}$
- $E_i \otimes \omega_X \in \mathcal{A}[1-\dim X]$
- $\nexists \underset{\neq 0}{F} \in \mathcal{A}' = \mathcal{A} \cap \mathcal{Z}$ s.t. $Z(F) = 0$.

Then, $\sigma' = (\mathcal{A}', Z' = Z|_{K(\mathcal{Z})})$ is a stability condition on $\mathcal{Z}$.

(Dropping $Z(F) \neq 0$ cond. gives weak.)

For $\mathrm{Ku}\, Y$, Y cubic fourfold, consider $\sigma_{s,q}$



$\mathcal{O}_Y \in \mathrm{Coh}^s(Y)$

$\mathcal{O}_Y$ is slope stable w/ $\mu_{slope} = 0$

need $s < 0$

$\mathcal{O}_Y(-3)$ slope stable with $\mu_{slope} = -3$

need $-3 \leq s \leq 0 \rightsquigarrow \mathcal{O}_Y(-3)[1] \in \mathrm{Coh}^s$.

NO.

Geometric trick.

$\ell \subseteq Y$

a line not on a plane in Y.

$$\begin{tikzcd} & Bl_\ell Y \arrow[dl] \arrow[dr, "\pi"] & \\ Y \arrow[rr, dashed] & & \mathbb{P}^3 \end{tikzcd}$$

π is a conic fibration $\rightsquigarrow$ $\mathcal{B}$ sheaf of Clifford algebras.

$\mathcal{B}_0$ even part.

project from ℓ

$$D^b(Bl_\ell Y) \simeq \langle D^b(\mathbb{P}^3, \mathcal{B}_0), D^b(\mathbb{P}^3) \rangle$$

Kuz. for quadric fibrations.

Something noncommutative.

$\wr\!\!\mid$ Orlov

$\langle D^b(Y), D^b(\ell), D^b(\ell) \rangle$

$\cup\!\mid$

$\mathrm{Ku}\,Y$

Putting together these $\simeq$s,

$$D^b(\mathbb{P}^3, \mathcal{B}_0) \simeq \langle \underset{\substack{\wr\!\mid \\ \mathrm{Ku}\,Y}}{\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle.$$

odd part of $\mathcal{B}$

Some twists...

Exceptional.

Upshot: have reduced the dimension from 4 to 3.

Define tilt stability on $D^b(\mathbb{P}^3, \mathcal{B}_0)$.

$\sigma_{-1,q}$, small rotation of this tilt stability

q small restricts to a stability condition on $\mathrm{Ku}\,Y$.

Via a slight generalization of the criterion.

Max Q. Does the rationality question for Y have something to do with $\mathcal{B}$?

Moduli spaces. $f: X \to S$ sm, prop.

[Kuznetsov] $\mathcal{Z} \in D_{perf}(X)$ is S-linear if...

S-linear SODs.

Thm (Kuznetsov). Base change for SODs.

$$D_{perf}(X) = \langle T_1, \ldots, T_m \rangle$$

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ S' & \to & S \end{array}$$

$$D_{perf}(X') \simeq \langle T_1', \ldots, T_m' \rangle,$$

$$T_i' \simeq D_{perf}(S') \otimes_{D_{perf}(S)} T_i.$$

Ex. $\begin{array}{c} \mathcal{Y} \\ \downarrow \\ S \end{array}$ family of cubic fourfolds.

$$D_{perf}(\mathcal{Y}) \simeq \langle \mathrm{Ku}_{\mathcal{Y}}, \underset{\overset{\wr\wr}{D^b(S)}}{\mathcal{O}_{\mathcal{Y}}}, \underset{\overset{\wr\wr}{D^b(S)}}{\mathcal{O}_{\mathcal{Y}}(1)}, \underset{\overset{\wr\wr}{D^b(S)}}{\mathcal{O}_{\mathcal{Y}}(2)} \rangle$$

is S-linear.

$s \in S$ gives SOD of fiber.

[Lieblich] $E \in D(X)$ universally gluable if $\mathrm{Ext}^i(E, E) = 0$, $i < 0$.

$$\mathcal{M}_{pug}(\mathcal{Y}/S)(T) = \left\{ \begin{array}{c} \text{perfect uni. gluable} \\ \text{on } \mathcal{Y} \times_S T \end{array} \right\}.$$

Thm (Lieblich). $\mathcal{M}_{pug}(\mathcal{Y}/S)$ is an alg. stack locally of finite type.

[BLMNPS] $\mathcal{M}_{pug}(\mathrm{Ku}(\mathcal{Y})/S) \subseteq \mathcal{M}_{pug}(\mathcal{Y}/S)$.

$$\cup\!\mathsf{I}$$

$\mathcal{M}_{\underline{\sigma}}(\mathrm{Ku}(\mathcal{Y}), v)$ fiberwise semistable of class v.

$\underline{\sigma}$ family of stability conditions restricting to those constructed above.

<u>Thm</u> [BLMNPS]. $S = \operatorname{Spec} k$ for simplicity; need family for results.

① There is a good moduli space $\mathcal{M}_\sigma(\mathrm{Ku}(\mathcal{Y}), v) \to M_\sigma(\mathrm{Ku}(Y), v)$ which is a proper algebraic space.

② If $v \in K^0_{num}(\mathrm{Ku}\,Y)$ (Mukai lattice), $v^2 \geq -2$, v prim., σ v-generic, then $M_\sigma(\mathrm{Ku}(Y), v)$ is nonempty and is a proj. HK manifold of dim $v^2+2 \underset{def}{\sim} K3^{[n]}$.

[Addington–Thomas]. $\exists\, \lambda_1, \lambda_2 \in K^0_{num}(\mathrm{Ku}\,Y)$ forming a $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ lattice

and $\langle \lambda_1, \lambda_2 \rangle^\perp \cong H^4(Y, \mathbb{Z})_{prim}$.

<u>Cor</u>. $v = a\lambda_1 + b\lambda_2$, $(a,b) = 1$

$\rightsquigarrow$ 20-dim family of polarized HKs.