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Bridgeland stability conditions and geometry of hyperKähler manifolds.

1.

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Intro. HK geometry.

A HK manifold is a compact complex Kähler manifold X s.t. $\pi_1 X = 1$ and $H^0(X, \Lambda^2 T_X^*) \cong \mathbb{C}\eta$, η a symplectic form.

Focus on projective examples.

Exs. $\dim X = 2$. K3 surfaces

$\dim X > 2$. Four examples in different deformation classes.

[Beauville].

① S a K3 surface, $S^{(n)} = S^{\times n}/\mathfrak{S}_n$, unique symplectic form, but singular.

$S^{[n]}$ = Hilbert scheme of n points on S

$\dim S^{[n]} = 2n$.

These are $K3^{[n]}$-type.

② A abelian surface, $\Sigma : A^{[n+1]} \xrightarrow{\text{Hilbert-Chow}} A^{(n+1)} \xrightarrow{\text{sum}} A$,

then $\Sigma^{-1}(0) =: K_n(A)$, $\dim 2n$

generalized Kummer variety GK^n-type.

①' [Mukai, ..., Yoshioka]. S a K3 surface.

$H^*(S, \mathbb{Z}) \cong H^0 \oplus H^2 \oplus H^4$, Mukai pairing.

$$\langle (\alpha_0, \alpha_2, \alpha_4), (\beta_0, \beta_2, \beta_4) \rangle = \alpha_2\beta_2 - \alpha_4\beta_0 - \alpha_0\beta_4$$

+ weight 2 Hodge structure:

$\widetilde{H}^{2,0}(S) = H^{2,0}(S)$,

$\widetilde{H}^{1,1}(S) = \oplus H^{p,p}(S)$.

$H_{alg}(S,\mathbb{Z}) = \tilde{H}^{1,1}(S) \cap H^*(X,\mathbb{Z})$ algebraic Mukai lattice.

Ψ

v

H polarization of S

$M_H(v)$ = moduli space of H-Gieseker semistable sheaves on S.

$$v(F) = ch(F)\sqrt{td_S}.$$

<u>Thm</u>. Let $v \in H^*_{alg}(S,\mathbb{Z})$ s.t. $v^2 = \langle v,v\rangle \geq -2$ and v is primitive and positive (could be the Mukai vector of a sheaf).

And, H is a v-generic polarization, so semistable = stable.

Then, $M_H(v)$ is a projective HK manifold of dim v^2+2 deformation equivalent to $K3^{[n]}$.

<u>Rem</u>. Symplectic form. $[F] \in M_H(v)$.

$$T_{[F]}M_H(v) = \mathrm{Ext}^1(F,F)$$

$$\begin{array}{c} \mathrm{Ext}^1(F,F)\times \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^2(F,F) \\ \hphantom{\mathrm{Ext}^1(F,F)\times \mathrm{Ext}^1(F,F) \longrightarrow\ } \wr\| \\ \hphantom{\mathrm{Ext}^1(F,F)\times \mathrm{Ext}^1(F,F) \longrightarrow\ } \mathrm{Hom}(F,F) \\ \hphantom{\mathrm{Ext}^1(F,F)\times \mathrm{Ext}^1(F,F) \longrightarrow\ } \wr\| \text{ stability} \\ \hphantom{\mathrm{Ext}^1(F,F)\times \mathrm{Ext}^1(F,F) \longrightarrow\ } \mathbb{C} \end{array}$$

<u>Rem</u>. [Bayer-Macri] Can extend to moduli space of stable objects wrt BSCs.

(2') Mukai Lattice $H^{ev}(A,\mathbb{Z})$, $v^2 \geq 6$.

<u>Thm</u>. Under some assumptions, $M_H(v)$ is smooth,

$$a_v : M_H(v) \xrightarrow[(\det\circ\Phi_P)\times\det]{} A\times\hat{A}$$

is the Albanese morphism. And $a_v^{-1}(0) =: K_v(A)$ is HK of dim v^2-2 def. equivalent to GK^n.

(3) [O'Grady]

[Lehn – Sorger] S a K3, $v_0 \in H_{alg}(S,\mathbb{Z})$, primitive, $v_0^2=2$. $v=2v_0$. H v-generic.

<u>Ex</u> O'Grady. I_Z, Z two points, $v(I_Z)=(1,0,-1)$.

OG10. <u>Thm</u>. singular $M_H(v)$ has a symplectic resolution which is HK manifold of dim 10 <u>not</u> def. equivalent to the others.

(4) [O'Grady]. $v_0=(1,0,-1)$, $v=2v_0$,

$$\begin{array}{ccccc} \tilde{M} & \to & M_H(v) & \to & A\times\hat{A} \\ \cup & & \text{singular} & & \\ \tilde{K} & \to & K_v(A) & & \end{array}$$

OG6. 6 dim HK not equivalent to others, can compute Betti numbers.

Rem. A general projective deformation of the previous exs is not of this form.

Eg (2*) $(X,h) \underset{\text{def}}{\sim} (K_v(A), h')$

$$\begin{array}{ccc} v^\perp & \xrightarrow{\sim} & H^2(K_v(A), \mathbb{Z}) \\ \cap & & \\ H^{ev}(A,\mathbb{Z}) \simeq U^{\oplus 4} & & \end{array}$$

$$U = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

$b_2(X) = 7$

$\rightsquigarrow h^1(T_X) = 5$

moduli has 4-dimensions as one fixes an ample class.

But, $b_2(A) = 6 \rightsquigarrow 3$ dim moduli

Similar for K3.

Problem. Describe general HK manifold.

Cubic 4-folds. $Y \subseteq \mathbb{P}^5$ $|\mathcal{O}_{\mathbb{P}^5}(3)| / PGL(6)$ 20-dim

(1) Beauville - Donagi

$F(Y)$ = variety parametrizing lines in Y

HK manifold $\underset{\text{def}}{\sim} K3^{[2]}$

$$H^4(Y,\mathbb{Z})_{prim} \cong H^2(F(Y),\mathbb{Z})_{prim}$$

(2) [Lehn–Lehn–Sorger–van Straten] $Y \not\supset$ plane

$M_3(Y)$ irred. component in $\mathrm{Hilb}^{3t+1}(Y)$

~~the~~ of twisted cubic curves in Y

$Z(Y)$ contraction of divisor HK 8-fold $\underset{\text{def}}{\sim}$ $K3^{[4]}$

(3) [LSV] [Addington...

compactification of [LPZ]

fibration in intermediate

Jacobians $\overline{J(Y)} \underset{\text{def}}{\sim}$ OG10.

<u>Plan</u>. Study moduli spaces in a certain subcategory of $D^b(Y)$.

Modular description of known examples.

If time, GK^n.