Perry.

3.

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**Thm** (H–P). $\mathcal{C}$ connected geometric CY category $/\mathbb{C}$ ($\mathcal{C} \underset{\text{adm}}{\hookrightarrow} D(X)$, $X$ sm. prop $/\mathbb{C}$).

Let $v \in K_0^{top}(\mathcal{C})$ be Hodge. Fix

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook] \arrow[d, "\text{open}"] & s\mathcal{M}(\mathcal{C},v) \arrow[d, "\mathbb{G}_m\text{-gerbe}"] \\ M \arrow[r, hook] & sM(\mathcal{C},v) \end{tikzcd}$$

preserved by $\mathcal{A}ut^0(\mathcal{C})$.

$$\begin{tikzcd} \mathcal{A}ut^0(\mathcal{C}) \arrow[d, "\mathbb{G}_m\text{-gerbe}"] \\ Aut^0(\mathcal{C}) =: G \end{tikzcd}$$

$\underline{\underline{\text{IF}}}$, $[M/G]$ is DM,

then there exists a perfect obstruction thy $\varphi : F \to \tau^{\geq -1} L_{[M/G]}$

which is <u>symmetric</u>, i.e. $F \underset{\Theta}{\simeq} F^\vee[1]$ and $\Theta^\vee[1] \simeq \Theta$.

<u>Upshot</u>. $[M/G]^{vir} \in CH_0([M/G])$.

<u>Pf idea</u>. $\mathcal{E} \in \mathcal{C}_{\mathcal{M}}$ universal object. There is a natural map

$$\begin{tikzcd} HH^*(\mathcal{C}) \otimes \mathcal{O}_{\mathcal{M}}[+2] \arrow[r, "\text{action}"] & \mathcal{H}om_{\mathcal{M}}(\mathcal{E},\mathcal{E})[2] \arrow[d, phantom, "\wr"] & \\ & (\mathcal{H}om_{\mathcal{M}}(\mathcal{E},\mathcal{E})[1])^\vee \arrow[r, "\text{action dual}"] \arrow[dr, "\beta"] & (HH^*(\mathcal{C})[1])^\vee \otimes \mathcal{O}_{\mathcal{M}} \arrow[d] \\ \tau^{\leq 1}(HH(\mathcal{C})) \otimes \mathcal{O}_{\mathcal{M}}[2] \arrow[uu] \arrow[ur, "\alpha"] & & \tau^{\geq 0}(HH^*(\mathcal{C})[1])^\vee \otimes \mathcal{O}_{\mathcal{M}} \end{tikzcd}$$

$\beta \circ \alpha \simeq 0$ for dumb reasons.

$$q : \mathcal{M} \to [\mathcal{M}/G] = [M/G]$$

$q^* F$ = "cohomology" of $\alpha, \beta$ sequence

$$:= \mathrm{cofib}\left( \tau^{\leq 1}(\mathcal{H}\mathcal{H}^*(\mathcal{C})) \otimes \mathcal{O}_{\mathcal{M}}[2] \to \mathrm{fib}(\beta) \right).$$

So, want to descend $F$.

Use a little DAG.

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook, "\text{closed}"] \arrow[d] & \mathcal{M}^{der} \arrow[d] \\ {[\mathcal{M}/G]} \arrow[r, hook, "i"] & {[\mathcal{M}^{der}/G]} \end{tikzcd}$$

derived enhancement, canonical.

" ——— "

Actually classical.

Then, $i^* L_{[\mathcal{M}^{der}/G]} \to L_{[\mathcal{M}/G]}$ is an obstruction theory.

$$\left( q^* L_{[\mathcal{M}^{der}/G]} \to L_{\mathcal{M}^{der}} \to \tau^{\geq 0}\left( (\mathcal{H}\mathcal{H}^*(\mathcal{C})[1])^\vee \right) \otimes \mathcal{O}_{\mathcal{M}^{der}} \right)\Big|_{\mathcal{M}}$$

Related to $L_G$.

$$\left( \mathrm{Hom}_{\mathcal{M}}(\mathcal{E}, \mathcal{E})[1] \right)^\vee$$

Looks like $\beta$.

Then, formally symmetrize.

Def. $\mathcal{C}$ as above, $v \in K_0^{top}(\mathcal{C})$ is Hodge, $\sigma$ a $v$-generic stability condition (so semistable = stable).

$$\begin{array}{c} \mathcal{M}_\sigma(\mathcal{C}, v) \xhookrightarrow{\text{open}} s\mathcal{M}(\mathcal{C}, v) \\ \| \\ \mathcal{M} \end{array}$$

presented by $\mathcal{A}ut^\circ(\mathcal{C})$, as one checks. When $[M/G]$ is DM and proper, then

$$DT_\sigma(v) := \#^{vir}[M/\mathrm{Aut}^\circ(\mathcal{C})] = \int_{[M/\mathrm{Aut}^\circ(\mathcal{C})]^{vir}} 1 .$$

Thm (VHC criterion for CY3 categories). $\mathcal{C}$ CY3$/S$, $S$ variety $/\mathbb{C}$.

$v \in \Gamma(K_0^{top}(\mathcal{C}/S))$ s.t. $v_s$ are Hodge $\forall s \in S$.

Assume $\sigma \in \mathrm{Stab}(\mathcal{C}/S)$ a stab. condition relative to $S$.

Assume $0 \in S$ s.t.

1) $\sigma_0$ is $v_0$-generic,
2) $[M_{\sigma_0}(v_0)/\mathrm{Aut}^\circ(\mathcal{C}_0)]$ is DM and proper,
3) $DT_{\sigma_0}(v_0) \neq 0$.

$\}$ Open conditions $\Rightarrow$ extend to a family proper over base, $DT \neq 0$ everywhere, so there are objects.

Then, $v_s$ algebraic $\forall s \in S$.

<u>Thm</u> (H-P). $X_1$ ab. 3-fold, $\alpha_1 \in Br(X_1)$, $\mathrm{ind}(\alpha_1) \mid \mathrm{per}(\alpha_1)^2$.

<u>Pf.</u> Suffices to show there is the following:

a) Family $X \xrightarrow{f} S$ of ab 3folds with pts $0, 1 \in S$,

b) $\alpha \in Br(X)$,

c) $v \in \Gamma(K_0^{top}(X, \alpha / S))$, Hodge in fibers, $\mathrm{rank}(v_1) = \mathrm{per}(\alpha_1)^2$,

d) $\sigma \in \mathrm{Stab}(D(X, \alpha)/S)$,

such that

1) $(X, \alpha)_1 = (X_1, \alpha_1)$,

2) $\alpha_0 = 0$ in $Br(X_0)$,

3) $\sigma_0$ $v_0$ generic,

4) $[M_{\sigma_0}(v_0) / \mathrm{Aut}^0] = DM$

5) $DT_{\sigma_0}(v_0) \neq 0$.

$$\Downarrow$$

The moduli space is non-empty <u>everywhere</u>!

Pf. a) – c) satisfying 1) – 2) not so bad.

Choose $b_1 \in H^2(X, \mathbb{Z})$ s.t. $\exp\left(\frac{b_1}{n}\right) = \alpha$ in $Br(X_1)[n]$.

Put $X$ in a family so that $b_1$ becomes algebraic on $X_0$ in polarized ab. 3folds. Then, $\alpha_0 = 0$.

$v$ comes from $0$.

d) Thm (Bayer-Macri-Stellari, BLMNPS, +ε).

$(X, \alpha) \to S$ family of twisted ab. 3-folds $\Rightarrow \mathrm{Stab}((X,\alpha)/S) \neq \emptyset$.

3) Bridgeland's deformation theorem $\Rightarrow$ up to perturbing $\sigma$, we can assume $\sigma_0$ is $v_0$-generic.

4) Lemma. $A$ simple ab. 3fold, $v \in K_0^{top}(A)$ s.t. $\Delta(v) \neq 0$, $\sigma \in \mathrm{Stab}(A)$ which is $v$-generic. Then, $[M_\sigma(v)/\mathrm{Aut}(D(A))]$ is DM.

Eventually, $A$ is $X_0$, with fiddling to arrange $A$ simple.

Pf. $\mathrm{Aut}^\circ(D(A)) \simeq A \times \hat{A}$. $E \in D(A)$.

$G_E$ = Stabilizer of $E$

$= \{(x, L) \in A \times \hat{A} : t_x^*(E) \otimes L = E\}$.

Mukai $\dim_{G_E} \leq 3$ and if $= 3$, then $\Delta([E]) = 0$.

$G_E \hookrightarrow A \times \hat{A}$

so, by simplicity, $\dim G_E = 0$ or $3$. But $\Delta([E]) \neq 0$ since we assume $\Delta(v) \neq 0$.

Upshot: holds if $X_0$ is simple and $\Delta(v_0) \neq 0$.

5.) WTS $DT_{\sigma_0}(v_0) \neq 0$.

We have $(X_0, b_0, H_0)$. Can assume $H_0$ is a principal polarization, $b_0$ algebraic.

Can choose $v$ s.t. $ch(v_0) = \left(n^2, -nb_0, \frac{b_0^2}{2} - tH_0^2, 1\right)$, $t \in \mathbb{Z}$.
As $t \longrightarrow \infty$, $\Delta(v) \longrightarrow > 0$.

$$\Phi_p : D(X_0) \simeq D(X_0^\vee)$$

induces an iso on

$$\begin{array}{ccc} K_0^{top}(X_0) & \cong & K_0^{top}(X_0^\vee) \\ \wr\| \downarrow ch & & \wr\| \downarrow ch \\ H^{ev}(X_0, \mathbb{Z}) & \underset{\Phi_p}{\cong} & H^{ev}(X_0^\vee, \mathbb{Z}) \end{array}$$

$\Phi_p = \pm PD$

$$H^n(X_0, \mathbb{Z}) \underset{PD}{\cong} H^{6-n}(X_0, \mathbb{Z})^\vee \cong H^{6-n}(X_0^\vee, \mathbb{Z}).$$
$$\wr\| \text{ p.pol}$$
$$H^{6-n}(X_0, \mathbb{Z})$$

$$ch(\Phi_p(v)) = \left(1, -PD\left(\tfrac{b_0^2}{2}\right) + t\,PD(H^2), -n\,PD(b_0), -n^2\right).$$

Choose $L \in Pic(X_0^\vee)$ s.t. $-c_1(L) = -\frac{PD(b_0^2)}{2} + t\,PD(H^2)$.

So, $ch(L \otimes \Phi_p(v)) = ch(L) \otimes ch(\Phi_p(v))$

$$= (1, 0, -\beta, -m).$$

So, we win as long as $\beta$ is of type $(1,1,d)$, using compatibility of DT invariants with autoequivalences.