Perry.

2.

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$f: X \to S$, $D(X) = D_{perf}(X)$.

$D_{perf}(X) \simeq \langle \mathcal{C}, \mathcal{D} \rangle$

S-linear SODs.

Theory of base change.

Ex. ① $X \to S$ cubic 4fold $\rightsquigarrow Ku(X)$.

② $\alpha \in Br(X)$, $[P] = \alpha$, $\mathcal{C} = D^b(X, \alpha) \subseteq D^b(P)$ X-linear.

Often thinking of it as S-linear.

Notation. $E, F \in \mathcal{C}$, $\mathcal{H}om_S(E,F) := f_* \mathcal{H}om_X(E,F)$, all derived.

$$\mathcal{H}om_S(E,F) \in D_{qc}(S)$$

$X \xrightarrow{f} S$ sm. prop, this is in $D_{perf}(S)$

Def. Hochschild cohomology of $\mathcal{C}$ is def'd as follows.

$$\mathcal{HH}^*(\mathcal{C}/S) \simeq \mathcal{H}om_S(\mathrm{id}_S, \mathrm{id}_S) \in D_{qc}(S) \in D(S) \quad X/S \text{ sm. prop.}$$

$$\mathrm{id}_S \in \mathrm{Fun}_S(\mathcal{C},\mathcal{C}) \subseteq D(X \times_S X) \quad P \text{ projector}$$

$\mathcal{HH}^i(\mathcal{C}/S)$ = coh. sheaf in degree i

<u>HKR.</u> $X \xrightarrow{f} S$ char. 0

$$\mathcal{HH}^*(X/S) \cong \bigoplus H^i(X, \Lambda^j T_{X/S})^* \quad \text{or}$$

$$\mathcal{HH}(X/S) \simeq \bigoplus Rf_* \Lambda^j T_{X/S}[-j].$$

<u>Def.</u> $\mathcal{C}/S$ is connected if $\mathcal{HH}^i(X_T/T) = \begin{cases} 0 & i<0, \\ \mathcal{O}_T & i=0. \end{cases}$

<u>Ex.</u> $X \to S$ sm. prop. + geo. connected fibers,

$D^b(X,\alpha)$ connected $/S$.

$\mathcal{C} \subseteq D(X)$ S-linear admissible, X/S sm. prop.

$\Rightarrow$ relative Serre functor $S_{\mathcal{C}/S} \curvearrowright \mathcal{C}$.

$$\mathcal{H}om_S(E,F)^\vee \simeq \mathcal{H}om_S(F, S_{\mathcal{C}/S}(E)).$$

<u>Ex.</u> ① $S_{D(X)/S} = \left(- \otimes \omega_f [\text{rel. dim}]\right)$.

Same for $D(X,\alpha)$.

<u>Def.</u> $\mathcal{C}/S$ is CY_n over S if $S_{\mathcal{C}/S} \simeq [n]$, at least Zariski locally on S.

Now, over $\mathbb{C}$.

<u>Thm</u> (Moulinos). $\mathcal{C} \subseteq D(X)$ S-linear admissible, X/S sm prop

$\rightsquigarrow$ $K_0^{top}(\mathcal{C}/S)$ loc. system on S^{an},

which carries a wt 0 variation of Hodge structures s.t.

1) Fibers recover Blanc's $K_0^{top}(\mathcal{C}_s)$,

2) additive,

3) $K_0^{top}(D(X)/S)$ is $\pi_0(f_*^{top}\, \underline{KU}_X)$ with $\overset{\text{variation}}{+\text{Hodge}}$ structures

$$K_0^{top}(D(X)/S)_{\mathbb{Q}} \underset{VHS}{\cong} \bigoplus R^{2k} f_*^{an}\, \mathbb{Q}(k).$$

<u>Conj</u> (Variational integral Hodge conjecture for categories).

Pick $v \in \Gamma(K_0^{top}(\mathcal{C}/S))$, $o \in S(\mathbb{C})$ s.t. $v_o \in K_0^{top}(\mathcal{C}_o)$ is algebraic (image of $K_0(\mathcal{C}_o) \to K_0^{top}(\mathcal{C}_o)$), then $v_s \in K_0^{top}(\mathcal{C}_s)$ is algebraic $\forall s \in S$.

<u>Rem</u>. False in general. Interesting when true.

<u>Thm</u> (Lieblich, Toën–Vaquié). $\mathcal{C} \subset D(X)$ S-linear admissible, X/S sm. prop.

$$\mathcal{M}(\mathcal{C}/S) : (Sch/S)^{op} \longrightarrow Gpds$$
$$T \longmapsto \{ E \in \mathcal{C}_T : Ext^{<0}(E_t, E_t) = 0 \ \forall t \in T \}$$

is an alg. stack locally of finite type over S.

Open substacks.

① $s\mathcal{M}(\mathcal{C}/S) \underset{\text{open}}{\subseteq} \mathcal{M}(\mathcal{C}/S)$ locus of simple objects: $\mathrm{Hom}(E_t, E_t) \cong k(t)$ $\forall t \in T$.

gerbe over alg. space

② $v \in \Gamma(K_0^{top}(\mathcal{C}/S))$, $\mathcal{M}(\mathcal{C}/S, v)$ locus of objects of class v.

Lemma. Assume $0 \in S(\mathbb{C})$ s.t. $v_0 = [E_0]$, $E_0 \in \mathcal{C}_0$ where $\mathrm{Ext}^i(E_0, E_0) = 0$, $i < 0$, and $\mathcal{M}(\mathcal{C}/S, v) \longrightarrow S$ is smooth at E_0.

Then, v_s is algebraic $\forall s \in S$.

Pf. $\Rightarrow$ E_0 deforms to E_U over an étale nbd U of $0 \in S(\mathbb{C})$.

And, E_U extends to $\underset{E_X}{\overset{\|}{E}} \in \mathcal{C}$, (at least after resolving sings of S).

Now, look at $[E] \in \Gamma(K_0^{top}(\mathcal{C}/S))$ which agrees with v_0 at 0 and hence with v everywhere, since it's a loc. system. ∎

Thm (P). $\mathcal{C}$ connected CY2 category over S, $v \in \Gamma(K_0^{top}(\mathcal{C}/S))$ s.t. $v_s \in K_0^{top}(\mathcal{C}_s)$ is Hodge $\forall s \in S$. Then, $s\mathcal{M}(\mathcal{C}, v) \to S$ is smooth.

Application. IHC($K_0(X)$) holds for $X \in \mathbb{P}^5$ cubic 4fold.

Reduces to simples on K3 by specialization.

($\Rightarrow$ IHC(X) in this case.)

DT theory. Idea. $v \in \Gamma(K_0^{top}(\mathcal{C}/S))$. Pick $0 \in S(\mathbb{C})$. Want to do:

$$\#\{ E_0 \in \mathcal{C}_0 : [E_0] = v_0 \}.$$

Expected dimension? $\quad ext^1(E_0,E_0) - ext^2(E_0,E_0) = 0$ for $\mathcal{C}$ CY3.

Since they are dual.

So, # might make sense. Impose stability.

Problem. Usually not of expected dim, e.g. $Hilb^n(\underset{\substack{\uparrow \\ CY3}}{X})$.

So, we'll take a virtual count, defined by taking some cycle and taking its degree.

But, classical DT invariants are zero for ab. varieties. So, need to modify. Replace moduli space by its quotient by $X \times \hat{X}$, X ab. var. But, also twist.

Def. M DM stack, $M \longrightarrow S$ a morphism.

A perfect obstr. theory [redacted] over S is

$$\varphi : F \longrightarrow \tau^{\leq 1} L_{M/S} \in D^{[-1,0]}_{qc}(M)$$

s.t. 1) F is perfect of Tor-amplitude $[-1,0]$

2) $\mathcal{H}^0(\varphi) : \mathcal{H}^0(F) \cong \mathcal{H}^1(L_{M/S})$

$\mathcal{H}^{-1}(\varphi)$ surjective.

Thm (Behrend–Fantechi). POT There is a canonical "virtual fundamental class"

$$[M]^{vir}_{\varphi} \in CH_{\dim S + \underbrace{\mathrm{vdim}_{\varphi}(M/S)}_{= \chi(F)}}(M)$$

If $M \to S$ is proper and $\mathrm{vdim}_{\varphi}(M/S) = 0$, then

$$\#^{vir}_{\varphi_s} M_s := \int_{[M]^{vir}_{\varphi}} 1 \quad \text{is}$$

independent of $s \in S(\mathbb{C})_{reg}$. in $\mathbb{Q}$.

Toy example. $M = V(s) \hookrightarrow A$ smooth, zero locus of section of v.b. E over A. of rk r.

Not assuming [redacted] lci. Then is a perfect ~~complex~~ obst. thy. on M

s.t. $i_*[M]^{vir} = c_r(E)$.

$$\underbrace{E^\vee|_M \longrightarrow \Omega_A|_M}_{\text{take this for } F.} \longrightarrow \Omega_M \longrightarrow 0 \qquad (S = pt)$$

local model

$S = \mathrm{Spec}\ \mathbb{C}$.

$\mathcal{C} \in D(X)$ adm. S-linear, $X/\mathbb{C}$ sm. prop. <u>and</u> connected(!)

$\mathcal{A}ut(\mathcal{C}) \longrightarrow$ groupoids.

$T \longmapsto$ T-linear auto-equivalences of $\mathcal{C}_T$.

Algebraic stack loc. of finite type, as

$$\begin{array}{ccc} \mathcal{A}ut(\mathcal{C}) & \xhookrightarrow[\text{open}]{} & s\mathcal{M}(\mathrm{Fun}_S(\mathcal{C},\mathcal{C})). \\ \Big\downarrow {\scriptstyle \mathbb{G}_m\text{-gerbe}} & & \bigcap {\scriptstyle \text{open}} \\ & & s\mathcal{M}(D(X \times_S X)) \\ \mathrm{Aut}(\mathcal{C}) \text{ algebraic space} & & \\ \cup\!\! \mid & & \\ \mathrm{Aut}^\circ(\mathcal{C}) \text{ connected component of id.} & & \\ \uparrow {\scriptstyle \mathbb{G}_m\text{-gerbe}} & & \\ \mathcal{A}ut^\circ(\mathcal{C}) & & \end{array}$$

$$\mathcal{M} \hookrightarrow s\mathcal{M}(\mathcal{C}, v)$$

$\circlearrowleft$

$\mathcal{A}ut^\circ(\mathcal{C})$

$$[\mathcal{M} / \mathcal{A}ut^\circ(\mathcal{C})]$$

Want to construct a perf. obs. thy.