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The period-index problem.

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The period-index conjecture.

k a field. A central simple k-algebra is a f.dim. associative k-algebra with center k and no non-trivial two-sided ideals.

Rem. 1) $A \cong M_n(D)$ for D division

2) A CSA $\Leftrightarrow$ $A \otimes_k \bar{k} \cong M_n(\bar{k})$.

Brauer group of k is $\{\text{CSA}/k\}/\sim$, $A \sim B \Leftrightarrow M_n(A) \cong M_m(B)$ for some $m,n \geq 1$.

$$\{\text{CSA}/k\}/\sim \;\cong\; \{\text{central f.dim division alg}/k\}/\cong$$

torsion abelian group.

Exs ① $Br(\bar{k}) = 0$.

② $Br(\mathbb{R}) \cong \mathbb{Z}/2 \cdot [\mathbb{H}]$.

③ $Br(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$.

④ $\underbrace{Br(\bar{k}(C)) = 0}_{\text{Tsen}}$.

⑤ $Br(\mathbb{C}(x,y))$ huge. Uncountable.

Def. $\alpha \in Br(k)$, $per(\alpha)$ = order of α in $Br(k)$.

$ind(\alpha) = \sqrt{\dim_k D}$, $\alpha = [D]$, D division.

Lemma. $per(\alpha) \mid ind(\alpha)$ and they have the same prime factors.

Conj. $X/k = E$, $\alpha \in Br(k(X))$, $ind(\alpha) \mid per(\alpha)^{d-1}$, $d = \dim X$.

Best possible.

Rem 1) Vacuous if $\dim X \leq 1$ (Tsen).

2) True if $\dim X = 2$ (de Jong, Lieblich).

3) Wide open for every field $k(X)$ with $d \geq 3$.

Not known that there is even any upper bound for fixed $k(X)$.

These lectures: $/\mathbb{C}$. $X/\mathbb{C}$ sm. proj.

$$Br(X) := \Big\{ \underset{\substack{\uparrow \\ \text{families of CSAs.}}}{\text{Azumaya alg } A \text{ over } X} \Big\} / \sim$$

$A \sim B \iff A \otimes End(P) \cong B \otimes End(Q)$, P, Q v. bundles

1) $Br(X) \hookrightarrow Br(k(X))$ with image the subgroup of unramified classes.

2) $Br(X) \cong H^2_{et}(X, \mathbb{G}_m)$.

$$\underbrace{(\mathbb{Q}/\mathbb{Z})^{b_2-\rho}}\quad 3)\ 0 \longrightarrow \frac{H^2(X,\mathbb{Q})}{H^1(X,\mathbb{Z}) + Pic(X)_{\mathbb{Q}}} \xrightarrow{\exp} Br(X) \longrightarrow H^3(X,\mathbb{Z})_{tors} \longrightarrow 0 \quad \text{exact.}$$

4) $\left\{ \begin{array}{c} \text{Az. alg } A \\ \deg(A) = n \end{array} \right\} \cong H^1(X, PGL_n)$

$\cong \left\{ \begin{array}{c} \text{twisted forms} \\ \text{of } \mathbb{P}^{n-1} \end{array} \right\}$

Saveri – Brauer varieties

<u>Some lower bounds</u>. Think about the Hodge theory of Sevari–Brauer varieties.

(de Jong – Perry)

<u>Lemma</u>. $X/\mathbb{C}$ sm. proj, $\alpha \in Br(X)$ is topologically trivial,

so lift α to $B \in H^2(X, \mathbb{Q})$, write $B = \frac{b}{n}$, $b \in H^2(X, \mathbb{Z})$,

assume $n\alpha = 0$.

Choose $P \xrightarrow[r]{\text{rel dim}} X$ SB, $[P] = \alpha$.

$$\begin{array}{c} P \\ \pi \downarrow \\ X \end{array}$$

Then, there is a class $h \in H^2(P, \mathbb{Z})$ s.t.

1) h restricted to ~~a~~ geometric fiber P_x is hyperplane class $\mathcal{O}(1) \in H^2(P_x, \mathbb{Z})$,

2) $\overset{nh + \pi^* b}{nh + b} \in H^2(P, \mathbb{Z})$ is algebraic,

3) $\bigoplus_j H^{d-2j}(X, \mathbb{Q})(j) \xrightarrow[\pi^* \cup (h + \pi^* B)^j]{} H^d(P, \mathbb{Q})$, iso of Hodge structures.

<u>Pf</u>. 1) + 2) $\Rightarrow$ 3) by Leray – Hirsch since $nh + b$ is algebraic, so it's a map of Hodge structures.

CHECK THIS!

$\mathcal{O}_P(1)$ is α-twisted on P, L α-twisted top. line bundle on X

$\mathcal{O}_P^{top}(1) := \mathcal{O}_P(1) \otimes L^{-1}$, $h = c_1(\mathcal{O}_P^{top})$

actual top line bundle

Taking nth powers $\Rightarrow$ nh is algebraic...

Cor. If $\operatorname{ind}(\alpha) \mid e$ (where $e < r$), then there exist classes $c_i \in H^{i,i}(X, \mathbb{Q})$ s.t.

$$\underbrace{(h+B)^e + c_1 (h+B)^{e-1} + \cdots + c_m (h+B)^{e-m}}_{= [\bar{L}]} \in H^{2e}(P, \mathbb{Z})$$

where $m = \min\{e, \dim X\}$.

Pf. $\operatorname{ind}(\alpha) \mid e \iff \exists$ twisted linear subspace $L \subseteq P_K$ of codimension e, $K = k(X)$

$L_{\bar{K}} \subseteq \mathbb{P}^r_{\bar{K}} \cong \mathbb{P}^r$ is a linear subspace.

CHECK THIS.

$\gamma = [\bar{L}] \in H^{e,e}(P, \mathbb{Z})$. $\blacksquare$

Use that integrally (because of top. triviality)

$$\bigoplus H^{d-2j}(X, \mathbb{Z}) \xrightarrow{\cong} H^d(P, \mathbb{Z}).$$

Ex. $\dim X = 3$, $\operatorname{ind}(\alpha) \mid n$, $\alpha \in \operatorname{Br}(X)[n]$.

$$\left(h + \tfrac{b}{n}\right)^n + c_1 \left(h + \tfrac{b}{n}\right)^{n-1} + c_2 \left(h + \tfrac{b}{n}\right)^{n-2} + c_3 \left(h + \tfrac{b}{n}\right)^{n-3} \in H^{2n}(P, \mathbb{Z}).$$

$$= h^n + \underbrace{(b + c_1)}_{①} h^{n-1} + \underbrace{\left(\left(\tfrac{b}{n}\right)^2 \binom{n}{2} + c_1 \cdot \frac{b(n-1)}{n} + c_2 \right)}_{②} h^{n-2} + (\cdots + c_3)\, h^{n-3}$$

① c_1 integral, $c_1 \in H^{1,1}(X, \mathbb{Z})$.

OK, choose $c_3 = - \ldots$

② $b^2(n-1) + 2(n-1)bc_1 + \underbrace{2nc_2}_{\text{integral}} \in 2n H^4(X, \mathbb{Z})$.

Conclusion: $\exists\, c \in H^{1,1}(X, \mathbb{Z})$, $d \in H^{2,2}(X, \mathbb{Z})$ s.t.

$$(n-1) b^2 + 2bc + d \equiv 0 \mod 2n.$$

Ex. (X, H) a v.gen. ppabelian 3-fold.

$$H^1(X,\mathbb{Z}) \cong \mathbb{Z}\cdot\{x_1,x_2,x_3,y_1,y_2,y_3\}$$

$$H^2 \ni H = \sum_{i=1}^{3} x_i y_i$$

$$b = x_1 y_2 + x_2 y_3 \in H^2(X,\mathbb{Z})$$

$$\alpha \text{ "=" } \frac{b}{n} \in \mathrm{Br}(X)[n]$$

Then, $\mathrm{ind}(\alpha) \nmid n$. (In fact, it is n^2 if n is prime.)

Recovers Kresch.