Categorical spectra with adjoints

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There is some interest in mapping out of categories with adjoints:

- TQFT's are symmetric monoidal functors $\mathbf{Bord}^X \to \mathcal{D}$.
- 6-functor formalisms are lax functors from $\mathbf{nCorr}(\mathcal{C}) \to \mathcal{D}$.

We can also take $n \to \infty$.

Definition 1. The walking adjunction is the 2-category **Adj** with morphisms $l\colon X \leftrightarrows Y : r$, 2-cells $\eta\colon \mathrm{id}_X \to rl$ and $\varepsilon\colon lr \to \mathrm{id}_Y$, subject to the relations $(l.\varepsilon)(\eta.l) = \mathrm{id}_l$ and $\mathrm{id}_r = (\varepsilon.r)(r.\eta)$.

Theorem 2 (Riehl-Verity)

The functor $\mathbb{D}^1 \to \mathbf{Adj}$ is an epi in $\infty\mathbf{Cat}$.

Remark 3. Univalence is crucial for the theorem, essentially because the uniqueness of adjoints is only given up to unique isomorphism. As a minimal counterexample, consider the valent 2-category **cat** of strict categories, strict functors, and natural transformations, where an adjunction is an adjunction in the classical sense. Let $\mathcal{E}$ be the walking isomorphism, and consider the functor swap: $\mathcal{E} \to \mathcal{E}$. The two mappings $\{\mathrm{id}_\mathcal{E}, \mathrm{id}_\mathcal{E}\}, \{\mathrm{id}_\mathcal{E}, \mathrm{swap}\}\mathbf{Adj} \to \mathbf{cat}$ agree by precomposing with $\mathbb{D}^1 \to \mathbf{Adj}$, but they are not in the same mapping space of $\infty\mathbf{Cat}^{\mathrm{valent}}(\mathbf{Adj}, \mathbf{cat})$ (for they are not equal)

Definition 4. An ∞-category $\mathcal{C}$ has left adjoints if every diagram below has a lift.

$$\begin{array}{ccc} S^n\mathbb{D}^1 & \longrightarrow & \mathcal{C} \\ {\scriptstyle r}\downarrow & \nearrow & \\ S^n\mathbf{Adj} & & \end{array}$$

Remark 5. Theorem 2 explains that the lifts above are unique if they exist, so a category has left adjoints iff it is local with respect to $S^n\mathbb{D}^1 \xrightarrow{r} S^n\mathbf{Adj}$ for every $n \geq 0$. In particular, the full subcategory $\infty\mathbf{Cat}^{\mathrm{adj}} \hookrightarrow \infty\mathbf{Cat}$ is reflective.

Definition 6. $\mathbf{nSp}^{\mathrm{adj}} := \mathrm{colim}_{\mathrm{Pr}^L}\left(\mathbf{nCat}^{\mathrm{adj}} \xrightarrow{\Sigma} \mathbf{n+1Cat}^{\mathrm{adj}} \xrightarrow{\Sigma}\to \cdots\right)$.

That is, an object of $\mathbf{nSp}$ is a categorical spectrum whose 0-th entry is at most an n-category and (hence) the k-th is at most an $(n+k)$-category.

Example: $\mathbf{nCorr}(\mathcal{C}) \in \mathbf{1Sp}^{\mathrm{adj}}$

Example: $B^{\infty-n}(\mathbf{Bord}_n^X) \in \mathbf{0Sp}^{\mathrm{adj}}$ for any tangential structure X.

Theorem 7

Categorical spectra with adjoints are closed under the Gray tensor product:

$$\begin{array}{ccc} \mathbf{nSp} \otimes \mathbf{mSp} & \longrightarrow & (\mathbf{n}+\mathbf{m})\mathbf{Sp} \\ \downarrow & & \downarrow \\ \mathbf{nSp}^{\mathrm{adj}} \otimes \mathbf{mSp} & \dashrightarrow & (\mathbf{n}+\mathbf{m})\mathbf{Sp}^{\mathrm{adj}} \end{array}$$

Idea. Write $\mathbb{D}^1 \otimes \mathbb{D}^1 \to \mathbb{D}^1 \otimes \mathbf{Adj}$ as a pushout of $\mathbb{D}^1 \to \mathbf{Adj}$ and $S\mathbb{D}^1 \to S\mathbf{Adj}$ and perform some calculus of mates. □

Corollary 8

Let $X \to Y$ be a morphism in $\mathbf{nSp}^{\mathrm{adj}}$. There is a square

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & * \\ \downarrow & \swarrow & \downarrow & \simeq & \downarrow \\ * & \longrightarrow & Z & \longrightarrow & \Sigma X \\ & & \downarrow & \swarrow & \downarrow \\ & & * & \longrightarrow & \Sigma Y \end{array}$$

Idea. "$Z \in \mathbf{nSp}^{\mathrm{adj}}$" and "$Z$ is closed under extensions". □

Example : The cobordism hypothesis translates to an adjunction

$$\mathbf{0Sp} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{0Sp}^{\mathrm{adj}}$$

given by $S^{-n} \mapsto B^{\infty}\mathbf{Bord}_n^{\mathrm{fr}}$.

The extension $X, Y \mapsto Z$ produces the cobordism category with defects.

- limit description: known cells.
- colimit description: cobordism hypothesis with singularities.

In the following diagram, the top is genuinely $\mathbb{E}_1$, while the bottom is $\mathbb{E}_\infty$.

$$\begin{array}{ccc} & (\mathbf{CatSp}, \otimes) & \\ \swarrow & & \searrow \\ (\mathbf{0Sp}^{\mathrm{adj}}, \otimes) & & (\infty\mathbf{Sp}^{\mathrm{adj}}, \otimes) \\ \searrow & & \swarrow \\ & (\mathbf{0Sp}, \otimes) & \end{array}$$

The middle is conjecturally $\mathbb{E}_\infty$ as well:

Conjecture 8. $(\infty\mathbf{Sp}^{\mathrm{adj}}, \otimes)$ is $\mathbb{E}_\infty$.

Idea. Categorical spectra are defined by the colimit

$$\mathbf{CatSp}^{\mathrm{adj}} := \mathrm{colim}^L_{\mathrm{Pr}}(\mathbf{0Cat}_*^{\mathrm{adj}} \xrightarrow{\Sigma} \mathbf{1Cat}_*^{\mathrm{adj}} \to \mathbf{2Cat}_*^{\mathrm{adj}} \to \cdots).$$

The parametrization of this diagram is a functor $\mathbb{N} \to \mathrm{Pr}^L$, which trivially extends to $\mathbf{FinSet}^{\mathrm{inj}} \to \mathrm{Pr}^L$. Try to extend it further via $\mathbf{FinSet}^{\mathrm{inj}} \to \mathbf{FinVect}^{\mathrm{inj}}$, $n \mapsto \mathbb{R}^n$. Since the resulting functor $\mathbb{N} \to \mathbf{FinVect}^{\mathrm{inj}}$ is cofinal, the colimit of the extension is still $\mathbf{CatSp}^{\mathrm{adj}}$.

The extension to $\mathbf{FinVect}^{\mathrm{inj}} \to \mathrm{Pr}^L$ is related to a certain conjectured $O(n)$-action on ${}^{\mathrm{adj}}\mathbf{Cat}_{ast}$; this yields the $\mathbb{E}_\infty$-structure because the functor $\mathbb{E}_\infty(\mathrm{Pr}^L) \to \mathrm{Pr}^L$ detects colimits under $\mathbf{FinVect}^{\mathrm{inj}}$ (essentially because of Corollary 3.2.3.2 in HA). □

Remark 9. Let $\mathbf{Tang}_{n \subset n+k}$ be the k-category of n-dimensional tangles embedded in an $n+k$-dimensional cube. Loops in $\mathbf{Tang}_{n \subset n+k+1}$ are closed $(n+1)$-dimensional tangles embedded in an $(n+k+1)$-dimensional cube, so that there is an evident map $\mathbf{Tang}_{m \geq n} \to \Omega\mathbf{Tang}_{m \geq n}$

sending a tangle to sorry, but it is not an equivalence. This can be understood by defining the categorical *pre*spectra with adjoints as the oplax colimit $\mathbf{CatSp}^{\mathrm{adj}} := \mathrm{colim}^{\mathrm{oplax}}_{\mathrm{Pr}^L}(\mathbf{0Cat}^{\mathrm{adj}}_* \xrightarrow{\Sigma} \mathbf{1Cat}^{\mathrm{adj}}_* \to \cdots)$. The bordism category is the 'fibrant replacement' of this prespectrum, i.e. $\mathbf{Bord}_n = \mathrm{colim}_k \Omega^k \mathbf{Tang}_{n\subset n+k}$.

One formulation of the tangle hypothesis asserts an equivalence $\mathrm{Fun}^{\mathrm{exc}}(\mathbf{Mfld}^{\mathrm{sfr}}_n, \mathcal{S}) \simeq \mathbf{nCat}^{\mathrm{adj}}_*$ defined on representables by $\mathbb{R}^k \mapsto \mathbf{Tang}_{n\subset n+k}$. (Here $\mathbf{Mfld}^{\mathrm{sfr}}_n$ are 'manifolds with a solid framing' and $\mathrm{Fun}^{\mathrm{exc}}$ is asking for the maps to behave nicely with cut-and-paste along submanifolds.) There is an $O(n)$-action on $\mathbf{Mfld}^{\mathrm{sfr}}_n$, and hence on $\mathrm{Fun}^{\mathrm{exc}}(\mathbf{Mfld}^{\mathrm{sfr}}_n, \mathcal{S})$ and $\mathbf{nCat}^{\mathrm{adj}}_*$. The functor $\mathbf{FinVect}^{\mathrm{inj}} \to \mathrm{Pr}^L$ above sends $\mathbb{R}^n \mapsto \mathbf{nCat}^{\mathrm{adj}}_*$ and the $O(n)$-action on $\mathbb{R}^n$ to this action.