

Naruki Masuda.

16 March 2026.

FRG Workshop.

Categorical spectra and stability.

§0. Analogy.

classical	derived	categorical
Set	Ani	(\mathcal{P}, co) -cats
$(0,0)$ -cat	$(\infty, 0)$ -cat	
\times cartesian	\times	\boxtimes
$(1,1)$ -cat	$(\infty, 1)$ -cat	oriented cat (Sometimes Gray category)
abelian groups	grouplike \mathbb{E}_∞ -monoids ($\simeq Sp^{gr}$) Sp	Symmetric mon \mathcal{A}_∞ -cat categorical spectra
\mathbb{Z}	\mathcal{S}	$\mathbb{F} = \mathcal{B}^{\text{an}} \mathbb{F} \text{in} \mathbb{F}$
ab. cat.	stable $(\infty, 1)$ -cat	stable oriented cats

§1 Definition.

Def. $\text{Cat Sp} = \infty \text{ Sp} = \lim_{P \in \mathbb{R}} \left(\dots \rightarrow \infty \text{ Cat}_+ \xrightarrow[\Omega]{\Sigma} \infty \text{ Cat}_+ \right) \cong (X_n, X_n \cong \Omega X_{n+1})_{n \geq 0}$

$\int B^{\infty}$

$\text{Chan}(\infty \text{ Cat}_+) \cong \lim \left(\dots \rightarrow \text{Chan}(\infty \text{ Cat}_+) \xrightarrow[B]{\Omega} \text{Chan}(\infty \text{ Cat}_+) \right)$

$\cong \infty \text{ SMC}$

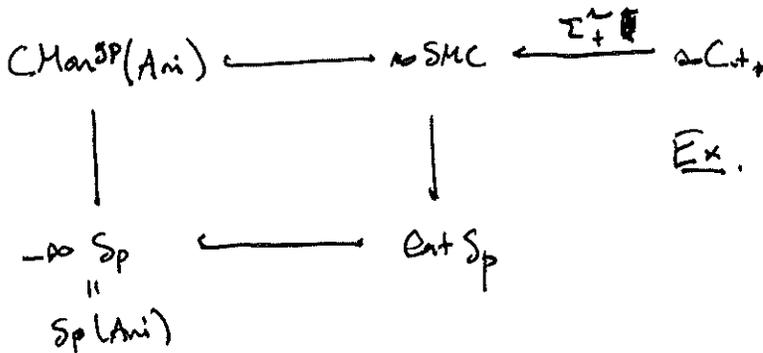
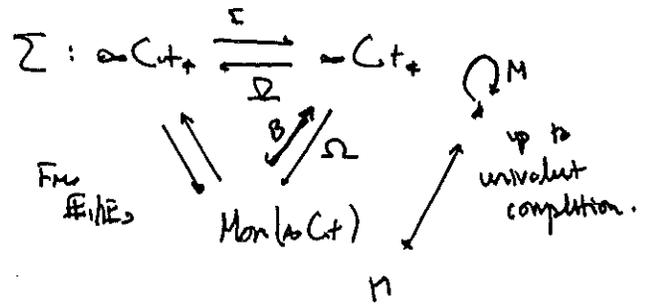
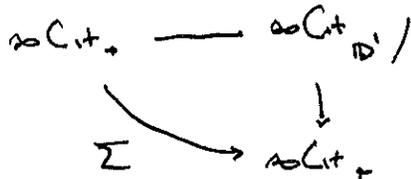
Rem. $k \text{ Sp} \subset \infty \text{ Sp} \quad k \in \mathbb{Z}$

ω

$(X_n) \leftrightarrow X_n \text{ max}(n+k, 0) \text{-cat.}$

$\text{Sp}(\text{Ani}) = -\infty \text{ Sp}$ Many interesting exs
 ex $\sim \text{Osp}, \text{Lsp}.$

Rem. $\Sigma X = \frac{SX}{S^*k}$



Ex. $\mathbb{F} = \Sigma^+_{\mathbb{F}}(\ast)$
 $= \mathbb{B}^{\infty} \text{Fin}^{\infty}.$

§2. Suspension and \boxtimes .

Prop.

$$\begin{array}{ccc}
 X \boxtimes \partial D' & \longrightarrow & X \boxtimes D' \\
 \downarrow & & \downarrow \\
 \partial D' & \longrightarrow & SX
 \end{array}
 \xrightarrow{(-)^{co}}
 \begin{array}{ccc}
 \partial D' \otimes X & \longrightarrow & D' \otimes X \\
 \downarrow & & \downarrow \\
 \partial D' & \longrightarrow & S(X^{coop}) \\
 & & \parallel \\
 & & S(X^{coop})
 \end{array}$$

Observation. $S: {}_{\infty}Cat \longrightarrow {}_{\infty}Cat_{\partial D'} \in \text{LMod}_{\infty Cat}(\text{Pr}^L)$.

Technical key. This upgrades to $\text{BMod}_{\infty Cat}(\text{Pr}^L)$ ← unital = LFT module is ∞Cat .
up to a twist.

∞Cat^{cop} right action twisted by $(-)^{coop}$, $\infty Cat \rightarrow \infty Cat$

$$(A, X, B) \longmapsto A \otimes X \otimes B^{coop}$$

Thm. $S: {}_{\infty}Cat \longrightarrow {}_{\infty}Cat_{\partial D'}^{cop}$ is a bimodule functor. (∃! in fact)

Cor. $\Sigma: {}_{\infty}Cat_+ \longrightarrow {}_{\infty}Cat^{cop}$ ~~is~~ BMod mor.

Redefn. $CatSp := \text{colim}$
 $\in \text{BMod}_{\infty Cat_+(\text{Pr}^L)} \left({}_{\infty}Cat \longrightarrow {}_{\infty}Cat^{cop} \longrightarrow {}_{\infty}Cat \longrightarrow \dots \right)$.

Rem. ${}_{\infty}Cat_+ \xrightarrow{\Sigma} {}_{\infty}Cat^{cop}$
 $\xrightarrow{\boxtimes} \vec{S}^1$ $\vec{S}^1 = \text{BIN} = \Sigma S^0$.

$$X \boxtimes Y = \frac{X \otimes Y}{X \vee Y}$$

$\vec{S}^2 \in \mathbb{Z}({}_{\infty}Cat_+^{\oplus})$.

Smulder arg \Leftarrow $\text{Ani} \xrightarrow{\quad} \text{Sp}(\text{Ani}) \in \text{Pr}^L$ idempotent
 $\therefore \text{Sp} = \text{Ani}[(\overline{S_1})^{-1}]$.

(1.2.3) $\square \text{S3}$.

Thm. $\text{Cat} \times \text{Sp} \in \text{BMod}_{\text{Cat}}(\text{Pr}^L)$ is idempotent

$\downarrow \Sigma_+$
 $\text{Cat} \boxtimes$

$\rightsquigarrow \exists ! \otimes$ on $\text{Cat} \times \text{Sp}$

promoting Σ_+ to a monoidal functor

\swarrow from both sides

$$\text{Cat} \times \text{Sp} \cong \text{Cat}_+ \otimes (\overline{S_1})^{-1}$$

This is "a node" of (bi)oriented categories.

Q. What is the center of (Cat, \boxtimes) ?

$\text{Cat} \boxtimes \text{Cat}^{\text{rev}}$ - centered.

$$\text{BMod}_{\text{Cat} \times \text{Sp}}(\text{Pr}^L) \xleftarrow{\quad} \text{BMod}_{\text{Cat} \times \text{Sp}}(\text{Pr}^L) \ni e \text{ in ess. m.}$$

$L \text{ or } R$ $L \text{ or } R$

$\Leftrightarrow e$ is pointed and $\overline{S_1}$ acts invertibly.

Def. A Gray (bi)module is stable if this

$\Leftrightarrow e$ pointed and

~~...~~ $\Sigma \rightarrow \Omega$ inv

Q. - Non-presentability?

- Stronger characterization?

Degeneration on stable (anl)-cats.

- $\emptyset \cong *$
 - $X \Delta Y \cong X \times Y$
 - $\Sigma \xrightarrow{\text{inv}} \Omega$
- } true in $\text{Cat} \times \text{Sp}$.
- $\text{cof } xy = \text{fib } xy$
 - $pa = \pi_i pb$
- } still false in $\text{Cat} \times \text{Sp}$

Exercise. TFAE for e $(\infty, 1)$ -cat.

① e is stable

② \forall finite I , $e \xleftrightarrow[\text{diag}]{\text{cat}_I} \text{Fun}(I, e)$
(both left adjoints)

③ $\forall J = \emptyset, 2$ points, $\begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix}$ can is true.
 \downarrow pointed \downarrow surjectivity \downarrow stability

Thm. $\text{Cat Sp} \xleftrightarrow[\substack{(-) \otimes (-) \\ (-)}$

$$\text{cat}_I \left(\begin{matrix} \text{Hom } X \rightarrow Z \\ \downarrow \\ \text{Hom } X \rightarrow Y \end{matrix} \right) \longleftrightarrow \left(\begin{matrix} X \rightarrow Z \\ \downarrow \\ Y \end{matrix} \right)$$