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Categorical spectra and stability.

§0. Analogy.

classical	derived	categorical	
Set	Ani	$(\cancel{\infty},\infty)$-cats	
$(0,0)$-cat	$(\infty,0)$-cat		
$\times$ cartesian	$\times$	$\boxtimes$	
$(1,1)$-cat	$(\infty,1)$-cat	oriented cat	(Sometimes Gray category)
abelian groups	grouplike $\mathbb{E}_\infty$-monoids ($\simeq Sp^{cn}$) Sp	Symmetric mon ∞-cat categorical spectra	
$\mathbb{Z}$	$\mathbb{S}$	$\mathbb{F} = \mathbb{B}^\infty \mathrm{Fin}^\simeq$	
ab. cat.	stable $(\infty,1)$-cat	stable oriented cats	

§1 Definition.

Def.

$$
\begin{array}{ccl}
\mathrm{Cat}\,\mathrm{Sp} = \infty\mathrm{Sp} & \underset{\mathrm{Pr}^R}{=} & \lim\left(\cdots \longrightarrow \infty\mathrm{Cat}_* \underset{\xleftarrow{\ \Sigma\ }}{\xrightarrow{\ \Omega\ }} \infty\mathrm{Cat}_*\right) \ni (X_n, X_n \simeq \Omega X_{n+1})_{n\geq 0} \\
\big\uparrow B^\infty & & \\
\mathrm{CMon}(\infty\mathrm{Cat}) & \simeq & \lim\left(\cdots \longrightarrow \mathrm{CMon}(\infty\mathrm{Cat}) \underset{\xleftarrow{\ B\ }}{\longrightarrow} \mathrm{CMon}(\infty\mathrm{Cat})\right) \\
\| & & \\
\infty\mathrm{SMC} & &
\end{array}
$$

Rem. $k\mathrm{Sp} \subset \infty\mathrm{Sp} \qquad k \in \mathbb{Z}$

$$\Downarrow$$

$(X_n) \Longleftrightarrow X_n \quad \max(n+k, 0)\text{-cat}.$

$\mathrm{Sp}(\mathrm{Ani}) = -\infty\,\mathrm{Sp}$ Hmy intuitivity exs as in $0\mathrm{Sp}$, $1\mathrm{Sp}$.

Rem. $\Sigma X = \dfrac{SX}{S*}$

$$
\begin{tikzcd}
\infty\mathrm{Cat}_* \arrow[r, no head] \arrow[dr, bend right, "\Sigma"'] & \infty\mathrm{Cat}_{\mathbb{D}^1/} \arrow[d] \\
& \infty\mathrm{Cat}_*
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Sigma : \infty\mathrm{Cat}_* \arrow[r, shift left, "\Sigma"] & \infty\mathrm{Cat}_* \arrow[l, shift left, "\Omega"] \arrow[dl, shift left, "\Omega"] & M \arrow[loop right] \\
\mathrm{Mon}(\infty\mathrm{Cat}) \arrow[u, shift left, "\mathrm{Free}/\mathrm{Forget}"] \arrow[ur, shift left, "B"] & & M \arrow[u, leftrightarrow, "\text{up to univalent completion.}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{CMon}^{\mathrm{gp}}(\mathrm{Ani}) \arrow[r, hook] \arrow[d] & \infty\mathrm{SMC} \arrow[d] & \infty\mathrm{Cat}_* \arrow[l, "\Sigma^\infty_+"'] \\
-\infty\,\mathrm{Sp} & \mathrm{Cat}\,\mathrm{Sp} \arrow[l] &
\end{tikzcd}
$$

$-\infty\,\mathrm{Sp} = \mathrm{Sp}(\mathrm{Ani})$

Ex. $\mathbb{F} = \Sigma^\infty_+(*) = \mathbb{B}^\infty \mathrm{Fin}^\simeq.$

§2. Suspension and $\boxtimes$.

Prop.

$$\begin{tikzcd} X \boxtimes \partial\mathbb{D}^1 \arrow[r] \arrow[d] & X \boxtimes \mathbb{D}^1 \arrow[d] \\ \partial\mathbb{D}^1 \arrow[r] & SX \end{tikzcd} \quad \overset{(-)^{co}}{\rightsquigarrow} \quad \begin{tikzcd} \partial\mathbb{D}^1 \otimes X \arrow[r] \arrow[d] & \mathbb{D}^1 \otimes X \arrow[d] \\ \partial\mathbb{D}^1 \arrow[r] & S(X^{co})^{co} \end{tikzcd}$$

$$S(X^{co})^{co} \cong S(X^{coop})$$

Observation. $S : \infty\mathrm{Cat} \longrightarrow \infty\mathrm{Cat}_{\partial\mathbb{D}^1/} \in \mathrm{LMod}_{\infty\mathrm{Cat}}(\mathrm{Pr}^L)$.

Technical key. This upgrades to $\mathrm{BMod}_{\infty\mathrm{Cat}}(\mathrm{Pr}^L)$ $\leftarrow$ underlying left module is $\infty\mathrm{Cat}$. up to a twist.

$\infty\mathrm{Cat}^{coop}$ right action twisted by $(-)^{coop}$, $\infty\mathrm{Cat} \to \infty\mathrm{Cat}$

$$(A, X, B) \longmapsto A \otimes X \otimes B^{coop}$$

Thm. $S : \infty\mathrm{Cat} \longrightarrow \infty\mathrm{Cat}^{coop}_{\partial\mathbb{D}^1}$ is a bimodule functor. ($\exists!$ in fact)

Cor. $\Sigma : \infty\mathrm{Cat}_* \longrightarrow \infty\mathrm{Cat}^{coop}$ BMod mor.

Redefn. $\mathrm{CatSp} := \operatorname{colim}\left(\infty\mathrm{Cat} \to \infty\mathrm{Cat}^{coop} \to \infty\mathrm{Cat} \to \cdots\right) \in \mathrm{BMod}_{\infty\mathrm{Cat}_*}(\mathrm{Pr}^L)$.

Rem. $\infty\mathrm{Cat}_* \longrightarrow \infty\mathrm{Cat}_*$
$- \boxtimes \vec{S}^1$ $\qquad \vec{S}^1 = B\mathbb{N} = \Sigma S^0$.

$$X \boxtimes Y = \frac{X \otimes Y}{X \vee Y}.$$

$$\vec{S}^2 \in Z(\infty\mathrm{Cat}_*^{\otimes}).$$

Similar say as $\mathrm{Ani} \longrightarrow \mathrm{Sp}(\mathrm{Ani}) \in \mathrm{Pr}^L$ idempotent

i.e. $\mathrm{Sp} = \mathrm{Ani}[(S^1)^{-1}]$.

$(1\ 2\ 3) \circlearrowright S^3$.

<u>Thm</u>. $\mathrm{CatSp} \in \mathrm{BMod}_{\infty\mathrm{Cat}}(\mathrm{Pr}^L_\omega)$ is idempotent

$\infty\mathrm{Cat}^{\boxtimes} \xrightarrow{\Sigma^\infty_+} \mathrm{CatSp}$

$\rightsquigarrow$ $\exists!$ $\otimes$ on CatSp

promoting Σ^∞_+ to a monoidal functor

from both sides

$$\mathrm{CatSp} \simeq \infty\mathrm{Cat}_+^{\boxtimes}\left((\vec{S}^1)^{-1}\right).$$

This is "a mode" of (bi)oriented categories.

$\infty\mathrm{Cat}^{\boxtimes} \otimes \infty\mathrm{Cat}^{\boxtimes\,\mathrm{rev}}$-enriched.

<u>Q</u>. What is the center of $(\infty\mathrm{Cat}, \boxtimes)$?

$\mathrm{BMod}_{\mathrm{CatSp}}(\mathrm{Pr}^L)_{L\text{ or }R} \longleftarrow \mathrm{BMod}_{\mathrm{CatSp}}(\mathrm{Pr}^L)_{L\text{ or }R} \ni \mathcal{C}$ in ess. im.

$\iff \mathcal{C}$ is pointed and $\vec{S}^1$ acts invertibly.

<u>Def</u>. A Gray (bi)module is stable if this $\iff \mathcal{C}$ pointed and $\Sigma \dashv \Omega$ inv

<u>Q</u>. - Non-presentably?

- Stronger characterization?

Digression on stable $(\infty,1)$-cats.

- $\emptyset \simeq *$
- $X \sqcup Y \simeq X \times Y$
- $\Sigma \overset{\mathrm{inv}}{\rightleftarrows} \Omega$

} true in CatSp.

- lax cof seq $=$ lax fib seq
- oplax po $=$ lax pb

} still false in CatSp

<u>Exercise</u>. TFAE for $\mathcal{C}$ $(\infty,1)$-cat.

① $\mathcal{C}$ is stable

② $\forall$ finite J,
$$\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "\mathrm{diag}"'] \arrow[r, phantom, "\perp"] \arrow[r, bend left=50, "\perp"] & \mathrm{Fun}(J,\mathcal{C}) \arrow[l, shift right, "\mathrm{colim}"'] \end{tikzcd}$$
(both left adjoints)

③ $J = \emptyset$, 2 points, $\ulcorner$ same is true.

$\emptyset$ ↓ pointed; 2 points ↓ semiadditive; $\ulcorner$ ↓ stability

<u>Thm</u>.
$$\begin{tikzcd} \mathrm{Cat\,Sp} \arrow[rr, bend left=40, "\Sigma^{\infty-1}\left(\begin{smallmatrix} S^0 \to I \\ \downarrow \\ I^{op} \end{smallmatrix}\right) \otimes -"] \arrow[rr, phantom, "\perp"] & & \mathrm{Fun}(\ulcorner, \mathrm{Cat\,Sp}) \arrow[ll, "(-) \overrightarrow{\amalg}_{(-)} (-)"] \end{tikzcd}.$$

$$\mathrm{colim}\begin{pmatrix} & \{1\}\otimes X \longrightarrow Z \\ & \downarrow \\ \{0\}\otimes X \hookleftarrow & \square^1 \otimes X \\ \downarrow & \\ Y & \end{pmatrix} \longleftarrow\!\shortmid \begin{pmatrix} X \longrightarrow Z \\ \downarrow \\ Y \end{pmatrix}$$