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FRG Workshop.

Categorical spectra part 2.

Recap: stability of $(\infty,1)$-cat (ad hoc). — More robust: $J = \emptyset, ++, \Gamma$

$$W : J^{op} \longrightarrow An/Sp, \quad j \longmapsto *$$

Def. $V \in Alg(Pr^L)$, $W : J^{op} \longrightarrow V$ weight for colimit is absolute for V-enrichment if there is a left adjoint to colim is absolute

$$Fun(J,V) \underset{colim^W}{\overset{\longleftarrow}{\longrightarrow}} V \in RMod_V(Pr^L)$$

$W \otimes (-)$

$$\begin{pmatrix} & X \longrightarrow Z \\ & \downarrow \\ X \longrightarrow & D_+ \boxtimes X \\ \downarrow & \\ Y & \end{pmatrix}$$

stability = "fancy weighted" colimits are absolute.

$$W^L \longleftarrow\!\shortmid \; W$$

$$Fun(\Gamma, CatSp) \underset{colim^W}{\overset{\longleftarrow}{\longrightarrow}} CatSp$$

$$W = \begin{pmatrix} \mathbb{D}^1 \longleftarrow \{1\} \\ \uparrow \\ \{0\} \end{pmatrix}$$

Left adjoint $W^L = \Sigma^{\infty-1}\begin{pmatrix} S^0 \longrightarrow I \\ \downarrow \\ I^{op} \end{pmatrix}$

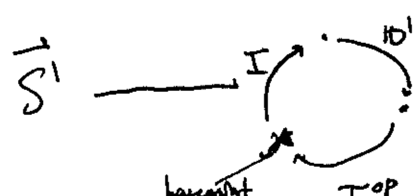
$$I = * \longrightarrow \bullet$$

This is an S-duality of profunctors/weights $W^L \dashv W$.

Unit $colim^W(W^L) = colim\, \Sigma^{\infty-1}\begin{pmatrix} & S^0 \longrightarrow I \\ & \downarrow \\ S^0 \longrightarrow & \mathbb{D}^1_+ \boxtimes S^0 \\ \downarrow & \\ I^{op} & \end{pmatrix}$

"$W \otimes W^L$"

$\mathbb{F}$



(**Ex.** $Fun(\Gamma, Sp) \underset{u}{\overset{\longleftarrow}{\longrightarrow}} Sp$) $\begin{matrix} S^1 \to 0 \\ \downarrow \\ 0 \end{matrix}$

I guess it's pushout, not binary coproduct.

Def. A left $\omega\mathrm{Cat}^{\boxtimes}$-enriched cat is "left stable" if it ~~then~~ is pointed and has $\vec{\square}$ and $\vec{\sqcup}$ is a right adjoint (or $\Sigma \dashv \Omega$ is invertible). ($\vec{\sqcup}$: oriented pushout)

Ex. $\mathrm{Cat} \hookrightarrow \omega\mathrm{Cat}\text{-}\mathrm{Cat}\text{-}\omega\mathrm{Cat}$.
$\quad = \mathrm{An}\text{-}\mathrm{Cat}\text{-}\mathrm{An}$

$\mathcal{C} \longmapsto \mathcal{C}$

$\mathcal{C}$ stable $\Longleftrightarrow$ $\mathcal{C}$ was stable to begin with.

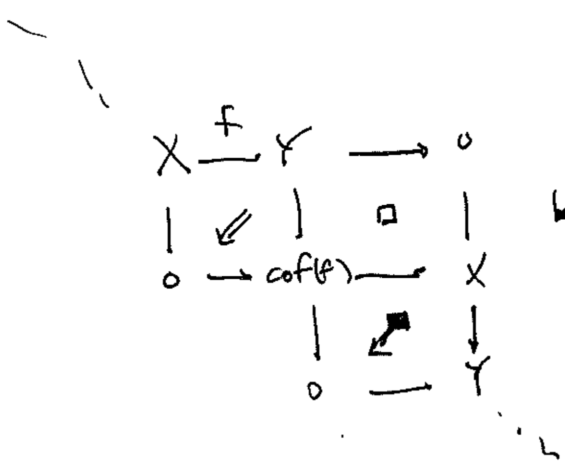
Cor. $\mathrm{Cat\,Sp} \underset{\vec{\sqcup}}{\overset{S^{-1}I^{op}}{\rightleftarrows}} \mathrm{Fun}(\ulcorner, \mathrm{CatSp}) \underset{\mathrm{RKE}}{\overset{\mathrm{res}}{\rightleftarrows}} \mathrm{Fun}(\rightarrow, \mathrm{CatSp})$

($S^{-1} \sqcup S^{-1} I \to S^{-1}I^{op}$; composite: $\Sigma^{\infty-1}(S^0 \to I)$)

$\vec{\mathrm{cof}}(f) \longleftarrow \begin{array}{c} X \xrightarrow{f} Y \\ \downarrow \\ 0 \end{array} \longleftarrow X \xrightarrow{f} Y$

wt of this is $S^0 \to I$ (or I^{op}?).

<u>Observation</u>. wt for oriented fiber. $\mathrm{colim}^{W} \simeq \lim^{W^L}$ becomes $\mathrm{cof}(f) \simeq \mathrm{fib}(\Sigma f)$.



bicartesian square,

Barratt–Puppe cofiber sequence.

$\rightsquigarrow$ Heine's categorical homology theory.

SW duality.

For spectra, $\mathcal{J} = *$, X finite

$$\mathrm{Sp} \underset{X}{\overset{X^L}{\rightleftarrows}} \mathrm{Sp}. \quad (\perp)$$

Thm*. TFAE for cat spectra:

- X is perfect (generated by $\mathbb{F}, 0, \Sigma^{-1}, \vec{\amalg}$, retracts);
- X is tiny: $[X, -] : \mathrm{CatSp} \to \mathrm{CatSp}$ commutes with weighted colimits; (CatSp)
- X is dualizable wrt $\otimes$.

§3. Cat. spectra w/adjoints.

TQFT : $\mathrm{Bord}_n \longrightarrow D$ $\quad\otimes$

6FF : $(n)\mathrm{Corr}(\mathcal{C}) \longrightarrow D$ $\quad\otimes$ lax

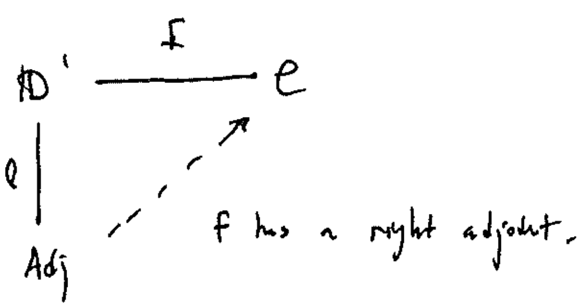
Recall. $\mathcal{C} \in \omega\mathrm{Cat}$.

adjunction in $\mathcal{C}$ = adjunct in $\mathrm{Ho}_2(\mathcal{C})$.

$\Longleftrightarrow$ $\mathrm{Adj} \longrightarrow \mathcal{C}$

walking adjunction cat. $x \underset{r}{\overset{l}{\rightleftarrows}} y$ $\qquad \mathrm{id} \xrightarrow{\eta} rl$, $\quad lr \xrightarrow{\varepsilon} \mathrm{id}$

Δ identity and coherence.



Thm (Riehl–Verity, folklore).

$\mathbb{D}^1 \longrightarrow Adj$ epi in ωCat (enough to show 2Cat). (Univalent setting.)

Add Haugseng.

Def. An ∞-cat has left adj for k-morphisms

$$\Longleftrightarrow \quad \begin{array}{ccc} S^{k-1}\mathbb{D}^1 & \xrightarrow{\ \forall\ } & \mathcal{C} \\ \downarrow & \overset{\exists}{\nearrow} & \\ S^{k-1} Adj & & \end{array}$$

$\infty Cat^{n\text{-adj}} = \{ \mathcal{C} : \mathcal{C} \text{ has adj. for } k \leq n \text{ morphisms} \}$.

Def. $n Sp^{adj} = \lim \left(\cdots \to (n+k) Cat_*^{adj} \longrightarrow (n+k-1) Cat_*^{adj} \cdots \right)$.

$n \in \mathbb{Z}$. ($\uparrow$ kth level) As many adjoints as reasonable.

Exs.

- $\{ n Corr(\mathcal{C}) \}_n \in 0Sp^{adj}$ (or 1 dep. on conv.)
- $B^{\infty-n} Bord_n^{\times k} \in 0Sp^{adj}$ ($\times k$: tangential structure)

Thm.

$$\begin{array}{ccc} nSp \otimes mSp & \longrightarrow & (n+m)Sp \\ \downarrow & & \downarrow \\ nSp^{adj} \otimes mSp^{adj} & \underset{\exists}{\longrightarrow} & (n+m)Sp^{adj} \end{array} \quad \Longrightarrow \quad 0Sp^{adj} \text{ has a localized tensor product.}$$

idea. Write $\mathbb{D}^1 \otimes \mathbb{D}^1 \longrightarrow \mathbb{D}^1 \otimes Adj$ as a pushout "mate calculus"

of $\mathbb{D}^1 \to Adj$ and $S\mathbb{D}^1 \to SAdj$.

Cor.

$$\begin{array}{ccccc} X & \to & Y & \to & 0 \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ 0 & \to & Z & \to & \Sigma X \\ & & \downarrow & \nearrow & \downarrow \\ & & 0 & \to & \Sigma Y \end{array}$$

If $Y, \Sigma X$ are in Sp^{adj}, then $Z \in Sp^{adj}$.

"Closed under extensions."

Ex. Cob. hyp. translates to

$$OSp \rightleftarrows OSp^{adj}$$

$$S^{-n} \longmapsto B^{\infty - n} Bord_n^{fr}$$

Similar for general tangential structures.

Extension $X, Y \rightsquigarrow Z$ produces cobordism cat. w/defects.

- lim description: know cells.
- colim description $\rightsquigarrow$ cob. hyp w/ singularities.

$$\begin{array}{ccc} & (CtSp, \otimes) \quad \mathbb{E}_1 & \\ \swarrow & & \searrow \\ (OSp^{adj}, \otimes) & & (\infty Sp^{adj}, \otimes) \\ \searrow & & \swarrow \\ & (Sp, \otimes) \quad \mathbb{E}_\infty & \end{array}$$

(Masuda–Rutter)

Conj. $\mathbb{E}_\infty$. $\longrightarrow (\infty Sp^{adj}, \otimes)$

Conditional arg.

$OSp^{adj} = \operatorname{colim}_{\mathbb{N}} \left(0Ct_+^{adj} \to 1Ct_n^{adj} \to \cdots \right)$

CONJ. Lifts to sym. mon. Functor

Get $\mathbb{E}_\infty$. $\quad$ $\begin{array}{ccc} \text{Fin Vect}^{adj} & \longrightarrow & Pr^L. \\ \uparrow & \mathbb{R}^n & \\ & \downarrow & \text{cofinal} \\ \mathbb{N} & n & \end{array}$