Hotchkiss.

3.

22 October 2023.

Complex analytic Brauer groups of tori.

Last time: stated that per-ind holds for 3-dim AVs.

M complex manifold. Holomorphic Azumaya algebras, etc. $\mathrm{Br}^{an}(M) \subseteq H^2(X, \mathcal{O}_X^\times)_{tors}$.

Lemma (Antieau–Williams). $\mathrm{per}(\alpha) \mid \mathrm{ind}(\alpha)$, same prime factors.

Tori. $X = \mathbb{C}^g/\Lambda$.

Thm (Huybrechts–Schröer). X torus with $g=2$ or an analytic K3, then $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$.

Pf. The space of complex tori is covered by twistor lines. Starting with (X_0, ω_0) Kähler, get $X(\omega) \to \mathbb{P}^1$ where all ~~[illegible]~~ fibers are smooth. $X(\omega) \ni X_0$

A v.b. on E_0 rarely extends over $X(\omega)$. E.g., $\det(E_0) \neq 0$, but $NS(X_t) = 0$ for most t. (This is the only obstruction.)

Verbitsky. If E_0 is slope stable (wrt ω_0), then $\mathbb{P}(E_0)$ does extend as a S-B manifold P over $X(\omega)$.

Now, $\det(E_0) \rightsquigarrow \Theta_0 \in H^2(X_0, \mathbb{Z}/\mathrm{rank}) \underset{\text{parallel transport}}{\simeq} H^2(X_t, \mathbb{Z}/\mathrm{rank}) \to \mathrm{Br}(X_t)$ goes to the class of P_t.

Proof of theorem. Go backwards and deform to a spot where the parallel transported Brauer class vanishes, putting the class in the image of NS.

Higher dimensional tori.

$X = \mathbb{C}^g/\Lambda$, $g \geq 3$.

X general, i.e., $NS(X) = 0$.

$Br^{an}(X)[n] \cong H^2(X, \mathbb{Z}/n)$.

$\alpha = \sum_{i=1}^{g} a_i x_i \wedge y_i$ $\quad$ ($\{x_i, y_i\}$ symplectic basis for $H^1(X,\mathbb{Z})$)

$a_i \in \mathbb{Z}/n$

$\mathbb{Z} \cdot a_i^{\perp} = \mathrm{ann}_{\mathbb{Z}}(a_i)$ (should just be the orders)

Practice bound. $\mathrm{ind}(\alpha) \mid \prod a_i^{\perp}$. If the $a_i = 1$,

$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^g$.

Thm (H). If X is general, $\overset{(g\geq 3)}{}$ $\mathrm{ind}(\alpha) = \prod a_i^{\perp}$ for all $\alpha \in Br(X)$.

$\left\{ \begin{array}{l} NS(X) = 0, \\ Hdg^4(X, \mathbb{Q}) = 0 \end{array} \right.$ &

Cor. There are lots of α with $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)^g$.

(Also true with coherent index.)

Idea of proof. (Voisin, Verbitsky). On a general torus of dim $g \geq 3$, any v.b. has a flat connection $\Rightarrow$ $ch(E) = (\mathrm{rk}\, E, 0, 0, \dots)$.

If E is α-twisted, $B \in H^2(X, \mathbb{Q})$ a B-field for α, then

$$ch^B(E) = (\overset{N}{\overset{\|}{\text{rank}\, E}}, 0, 0, \dots)$$
$$= \exp(B) \cdot ch(v)$$
$$v \in K_0^{top}(X) \underset{ch}{\cong} H^{2*}(X, \mathbb{Z}).$$

~~But,~~ $ch(v) = N \cdot \exp(-B) \in H^{2*}(X, \mathbb{Z})$

$$\Rightarrow \prod a_i^{\perp} \mid N.$$

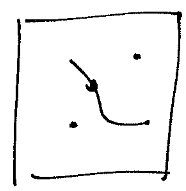
DT theory on abelian 3-folds.

Curve counts. (Bryan – Oberdieck – Pandharipande – Yin).

X ab 3-fold, $\beta \in H_2(X, \mathbb{Z})$, $n \in \mathbb{Z}$

$$\mathrm{Hilb}(\beta, n) = \{ \underset{\substack{\text{closed,} \\ \dim Z \leq 1}}{Z \subseteq X} : [Z] = \beta,\ \chi(\mathcal{O}_Z) = n \}.$$

X acts on $\mathrm{Hilb}(\beta, n)$ by translation



β

points contributing to $\chi(\mathcal{O}_Z)$.

Fact: X simple, then X acts with finite stabilizers and

$$[\mathrm{Hilb}(\beta,n)/X]$$

has a perfect obstruction theory. $\rightsquigarrow$ $\#^{vir}[\mathrm{Hilb}(\beta,n)/X]$.

$$\overset{!!}{}$$

Deformation invariant when β remains of Hodge type!!! ⟶ $DT(\beta, n)$

One example by hand. $X = J(C)$, C genus 3. (simple + pp $\Downarrow$)

Choose a point of C to embed $C \hookrightarrow X$.

Fact (Matsusaka). Any curve on X of class $[C]$ is C or $-C$ up to translation.

$$\underset{\chi(\mathcal{O}_C)}{\underset{\|}{DT([C],-2)}} = 2. \qquad \mathrm{Hilb}([C],-2)/X \simeq * \sqcup *.$$

Notation. $\beta \in H_2(X,\mathbb{Z})$ has type (a,b,c) if

$$\beta = a x_1 \wedge y_1 + b x_2 \wedge y_2 + c x_3 \wedge y_3 \in \Lambda^2 H_1.$$

$$DT(a,b,c;n) := DT(\beta,n).$$

Thm (BOPY, Oberdieck–Shen).

$$\sum_{n,d} DT(1,1,d;n) q^n t^d = (q+2+q^{-1}) \prod_{m\geq 1} \frac{(1+qt^m)^2(1+q^{-1}t^m)^2}{(1-t^4)^m}$$

Proof passes through GW theory and compares it to DT theory.

Rem. All coefficients are positive if $4d-n^2 \geq 0$.

What about non-curve classes?

σ stability condition

$v \in K_0^{top}(X)$

$$DT_\sigma(v) = \#^{vir}\left[M_\sigma(v) / X \times X^\vee \right]$$

if DM

Igusa

X abelian 3-fold

Fact (Oberdieck – Piyaratne – Toda). There is a quartic polynomial Δ defined on $K_0^{top}(X)_{\mathbb{Q}}$ satisfying:

1. Δ is invariant under derived autoequivalences.

2. If $ch(v) = (1, 0, -\beta, -n)$, then

   $\Delta(v) = 4d - n^2$

   looks like $\mathcal{O}_C$ for C a curve.

   if $\beta = (1,1,d)$ type

3. If $\Delta(v) > 0$, $[M_\sigma(v)/X \times X^\vee]$ is DM (X simple) and $DT_\sigma(v)$ does not depend on σ (so let it be denoted by $DT(v)$).

   (vanishing of wall crossing contribution)

So, if $g \in Aut(D^b(X))$,

$$DT(v) = DT_\sigma(v) = DT_{g\cdot\sigma}(g \cdot v) = DT(g \cdot v).$$

(Invariant under autoequivalences.)