Hotchkiss.

2.

21 October 2023.

$X/\mathbb{C}$ sm. proj. dim d. rank $(\mathrm{Hdg}(X,\alpha)) = \mathrm{ind}_H(\alpha)\cdot\mathbb{Z}$

$\alpha \in \mathrm{Br}(X)$

$\mathrm{per}(\alpha) \mid \mathrm{ind}_H(\alpha) \mid \mathrm{ind}(\alpha)$.

$D^b(X,\alpha)$

$K_0^{top}(X,\alpha)$

$\cup\!\!\mid$

$\mathrm{Hdg}(X,\alpha)$

Today: understand $\mathrm{ind}_H(\alpha)$ for threefolds.

α top trivial. $\mathrm{Hdg}(X,\alpha) \underset{\text{rank pres.}}{\cong} \left\{ v \in K_0^{top}(X) : \mathrm{ch}(v)\cdot\exp(B) \in \bigoplus_{i=0}^{d} H^{i,i}(X,\mathbb{Q}) \right\}$.

B a B-field for α.

<u>Cor of setup</u>. If $\mathrm{per}(\alpha)$ is prime to $(d-1)!$, then $\mathrm{ind}_H(\alpha) \mid \mathrm{per}(\alpha)^{d-1}$.

α not top. trivial. $\mathrm{rank}\left(K_0^{top}(X,\alpha)\right) = \mathrm{ind}(\alpha^{top})\cdot\mathbb{Z}$.

Example. $d = 3$

$$\Lambda = \mathrm{ch}\left(K_s^{top}(X) \right) \subseteq H^{ev}(X, \mathbb{Q}).$$

$\exp(-B) \cdot \exp(B) = \mathbb{1}$, which is Hodge.

Could multiply $\exp(-B)$ by some N to get in Λ.

Best thing James knows.

$n = \mathrm{per}(\alpha)$

$$\exp(-B) = \left(1, -B, \frac{B^2}{2}, -\frac{B^3}{6}\right)$$

$$\xi = \widetilde{2n^2 \exp(-B)} = 2\left(n^2, -n^2 B, n^2 \frac{B^2}{2}, 0\right).$$

$\xi \cdot \exp(B)$ is Hodge as its only off from $2n^2\mathbb{1}$ in deg 6, which is all Hodge anyways.

Claim: $\xi \in \Lambda$.

$$\Longrightarrow \mathrm{ind}_H(\alpha) \mid \mathrm{rank}(\xi) = 2n^2.$$

$$\underset{\text{period odd}}{\Longrightarrow} \mathrm{ind}_H(\alpha) \mid n^2.$$

If n is even, get $\mathrm{ind}_H(\alpha) \mid 8$ if $n = 2$.

$\underline{n=2}$.

- $\underline{\text{If}}$ $\mathrm{ind}_H(\alpha) = 2$, then there is $h \in \mathrm{Hdg}^4(X, \mathbb{Z})$ s.t. $b^2 + h \equiv 0 \mod 2$, when $b = 2B$, $b \in H^2(X, \mathbb{Z})$.

  This is Kresch's obstruction for $\mathrm{ind}(\alpha) = 2$ (since in fact $\mathrm{ind}_H(\alpha) \geq 4$).

- $\underline{\text{If}}$ $\mathrm{ind}_H(\alpha) = 4$, then there must be $s \in \mathrm{Hdg}^2(X, \mathbb{Z})$, $t \in \mathrm{Hdg}^4(X, \mathbb{Z})$, when $b^2 + sb + t \equiv 0 \mod 2$. $(\star)$

$\underline{\text{Key point}}$. If period-index holds for period 2^n unramified classes on threefolds, then $(\star)$ holds for all $b \in H^2(X, \mathbb{Z})$. (b not divisible by 2)

$\underline{\text{Application to IHC}}$.

By def, if $\mathrm{ind}_H(\alpha) < \mathrm{ind}(\alpha)$, then IHC fails for $D^b(X, \alpha)$. If in addition, $H^*(X, \mathbb{Z})$ is torsion free, then IHC fails for any SB P of class α. Indeed, $\mathrm{IHC}(D^b(P))$ fails, so $\mathrm{IHC}(P)$ fails.

$\underline{\text{Ex}}$ (Gabber). C genus ≥ 2, E_1, E_2 elliptic, $\omega_1, \omega_2 \in H^1(C, \mathbb{Z})$, $x_i \in H^1(E_i, \mathbb{Z})$. non zero, primitive, $\{\omega_i\}$ linearly independent.

$$B = \frac{1}{n}(\omega_1 \boxtimes x_1 + \omega_2 \boxtimes x_2) \in H^2(C \times E_1 \times E_2, \mathbb{Q})$$

$\rightsquigarrow \alpha$ has per $= n$, ind $= n^2$.

If $\omega_1 \cup \omega_2 = 0$, then $B^2 = 0$, so $v = (n, -nB, 0, 0) \in \mathrm{ch}(K_0^{top}(C \times E_1 \times E_2))$ and $v \cdot \exp(B) = (n, 0, 0, 0)$, which is Hodge. So, $\mathrm{ind}_H(\alpha) = n$.
$\Rightarrow \mathrm{IHC}(D^b(X, \alpha))$ fails.

DM surfaces.

Thm. If X is a smooth proper DM surface, then IHC holds for $D^b(X)$.

Rem. $\alpha \in \underset{\substack{\uparrow \\ \text{sm. proj. variety}}}{Br(X)}$, then an associated μ_n-gerbe $\mathcal{X}$ is a sm. prop. DM stack.

$$D^b(\mathcal{X}) \simeq \langle D^b(X), D^b(X,\alpha), \ldots \rangle.$$

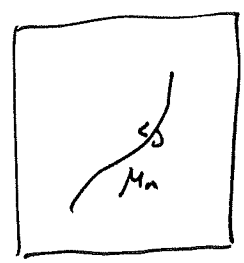
$$IHC(D^b(\mathcal{X})) \Leftrightarrow IHC(D^b(X, k\alpha)) \ \forall k.$$

Idea. Two ways to cook up new DM surfaces.

- X sm. proj. surface, $D \subseteq X$ sm. Cartier divisor, $n > 0$

  $X_{\frac{1}{n}D}$ root stack along D

  $$D^b(X_{\frac{1}{n}D}) \simeq \langle D^b(X), \underbrace{D^b(D), \ldots, D^b(D)}_{\substack{n-1 \\ \text{non-trivial characters}}} \rangle$$



  So, $IHC(D^b(X_{\frac{1}{n}D})) \Leftrightarrow IHC(D^b(X)) + \underset{\substack{\uparrow \\ \text{Always true.}}}{IHC(D^b(D))}$

  $$\begin{array}{ll} X & \\ \downarrow & G\text{-gerbe} \qquad G \text{ finite group} \\ S & \text{variety, orbifold} \end{array}$$

• $S = pt$, $X = BG$.

$X^\vee$ = moduli space of simple coherent sheaves on BG.

$$\simeq \underbrace{* \sqcup \cdots \sqcup *}_{\text{irreps}}$$

Universal sheaf on $X \times X^\vee$, $FM_\xi : D^b(X^\vee) \simeq D^b(X)$.

This is the rep theory of finite groups.

General case.

$$\begin{tikzcd} X \arrow[d, "\text{``}G\text{-gerbe''}"] \\ S \end{tikzcd}$$

$X^\vee$ relative moduli space of simple sheaves.

$X^\vee$ might not be fine, so

$$\xi \in D^b(X \times_S X^\vee, \underset{\substack{\uparrow \\ \text{pullback from } X^\vee.}}{\alpha})$$

$$FM_\xi : D^b(X^\vee, \alpha) \simeq D^b(X).$$

<u>Pf of Thm.</u> Reduce to case of $D^b(X, \alpha)$ when X is an iterated root stack over a sm. proj. surface.

Only thing to check: $\mathrm{ind}_H(\alpha) = \mathrm{ind}(\alpha)$. Reduce to de Jong.

Abelian varieties.

X ab. var. of dim g.

Alex: can find $\alpha \in Br(X)[n]$ s.t. $\mathrm{ind}(\alpha) = n, n^2, \ldots, n^{g-1}$.

Lemma. $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^g$.

Pf. Lift α to $\Theta \in H^2(X, \mathbb{Z}/n) \simeq \Lambda^2 H^1$,

$$\Theta = a_1 x_1 \wedge y_1 + \cdots + a_g x_g \wedge y_g,$$

$\{x_i, y_j\}$ symplectic basis for H^1.

As $\mathrm{ind}(\alpha) = \min\{\deg X' \to X \mid \alpha_{|X'} = 0\}$, and we can kill Θ by killing $x_1, \ldots, x_g$ via a degree n^g isogeny.

Thm. If $g = 3$, then $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^2$.

Pf. To come.

Rem. $\mathrm{ind}_H(\alpha) \mid \mathrm{per}(\alpha)^2$ for abelian threefolds by a simple calculation in $H^*(X, \mathbb{Z})$.

Strategy.

(1) Construct $v \in \mathrm{Hdg}(X, \alpha)$ with $\mathrm{rank}(v) = \mathrm{per}(\alpha)^2$.

(2) Show v is algebraic using Donaldson – Thomas theory.

A : Technical input.

J : Geometry.

A : Finish.