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The period-index problem.

[illegible]. 1

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Hodge theory of twisted derived categories.

Replace $H^n(-,\mathbb{Z})$ with $K^{top}(-)$.

$\mathcal{C}$ $\mathbb{C}$-linear Δed category. dg, ∞, etc.

Blanc: $\rightsquigarrow$ $K_0^{top}(\mathcal{C})$.

1. $\mathcal{C} = D_{perf}(X) \Rightarrow K_0^{top}(\mathcal{C}) \cong KU^0(X(\mathbb{C}))$.

   $\uparrow$ Grothendieck group of top. $\mathbb{C}$-vbs

2. $\mathcal{C} \simeq \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ SOD

   $\Rightarrow K_0^{top}(\mathcal{C}) \cong K_0^{top}(\mathcal{C}_1) \oplus K_0^{top}(\mathcal{C}_2)$.

<u>Def</u>. $\mathcal{C}$ is <u>geometric</u> if there is an admissible embedding $\mathcal{C} \hookrightarrow D^b(X)$, $X/\mathbb{C}$ sm. proj.

<u>Observation</u>. If $\mathcal{C}$ is geometric, then, its top. K-theory has a natural weight 0 Hodge structure s.t.

$$K_0^{top}(X)_{\mathbb{Q}} \cong \bigoplus_{i \geq 0} H^{2i}(X,\mathbb{Q})(i).$$

Now, use retracts.

Finally, $K_0(\mathcal{C}) \longrightarrow K_0^{top}(\mathcal{C})$ factors through $Hdg(\mathcal{C})$ of integral Hodge classes.

So, can formulate the integral Hodge conjecture for $\mathcal{C}$.

<u>Example</u>. $\mathcal{C} = D^b(X)$. This is a <u>version</u> of the integral Hodge conjecture for X.

Assume $H^*(X,\mathbb{Z})$ is torsion free. Then $IHC^*(X) \Rightarrow IHC(D^b(X))$.

In general, both versions are false.

We think there are examples where $IHC(D^b(X))$ but not $IHC^*(X)$.

Twisted derived categories.

X sm. proj, $\alpha \in Br(X)$, $\mathcal{A}$ Azumaya with $[\mathcal{A}] = \alpha$.

Modules over $\mathcal{A}$ $\rightsquigarrow$ $D^b(X,\mathcal{A}) =: D^b(X,\alpha)$. Căldăraru.

Remark. $D^b(X,\alpha)$ is geometric.

$$D^b(P) \simeq \langle D^b(X), D^b(X,\alpha), \ldots, D^b(X,(d-1)\alpha) \rangle$$

$K_0(X,\alpha) = K_0(D^b(X,\alpha))$, etc.

$$\begin{tikzcd}
K_0(X,\alpha) \arrow[rr] \arrow[dr] & & K_0^{top}(X,\alpha) \arrow[r, "\mathrm{rank}"] & \mathbb{Z} \\
& \mathrm{Hdg}(X,\alpha) \arrow[ur] & &
\end{tikzcd}$$

and $\quad \mathrm{rank}\left(K_0(X,\alpha)\right) = \mathbb{Z}\cdot \mathrm{ind}(\alpha) \subseteq \mathbb{Z}.$

<u>Strategy for period-index</u>.

1. Construct $v \in \mathrm{Hdg}(X,\alpha)$ of $\mathrm{rank}(v) = \mathrm{per}(\alpha)^{d-1}$, $d = \dim X$.

2. Lift v to $K_0(X,\alpha)$ $\left(\mathrm{IHC}(D^b(X,\alpha))\right)$.

<u>Def</u>. $\mathrm{ind}_H(\alpha)$ is such that $\mathrm{rank}\left(\mathrm{Hdg}(X,\alpha)\right) = \mathbb{Z}\cdot\mathrm{ind}_H(\alpha) \subseteq \mathbb{Z}$.

<u>Lemma</u>. $\overbrace{\mathrm{per}(\alpha) \mid \mathrm{ind}_H(\alpha)}^{\text{WHY?}} \mid \mathrm{ind}(\alpha).$

<u>Rem</u>. If $\mathrm{ind}_H(\alpha) < \mathrm{ind}(\alpha)$, then $\mathrm{IHC}(D^b(X,\alpha))$ fails.

Period-index conjecture $\Rightarrow$ $\mathrm{ind}_H(\alpha) \mid \mathrm{per}(\alpha)^{d-1}$.

Twisted Mukai structures.

$$H^2(X,\mathbb{Q}) \longrightarrow Br(X) \longrightarrow H^3(X,\mathbb{Z})_{tors} \longrightarrow 0.$$

α top. trivial
lifts to
$B \in H^2(X,\mathbb{Q})$.

<u>Thm</u> (H.). The Hodge structure on $K_0^{top}(X,\alpha)_{\mathbb{Q}}$ is given as follows:

- $K_0^{top}(X,\alpha) \cong K_0^{top}(X)$ as abelian groups (using top. triviality);

- $K_0^{top}(X)_{\mathbb{Q}} \cong \bigoplus H^{2i}(X,\mathbb{Q})(i)$

  $v \longmapsto ch(v)$ from before.

  Now, we twist:

  $$v \longmapsto ch(v)\cdot \exp(B) = ch(v)\left(1 + B + \frac{B^2}{2} + \frac{B^3}{6} + \dots\right)$$

  So, pullback along iso.

<u>Pf</u>. $P \to X$ SB of class α. Then, $K_0^{top}(P) \cong K_0^{top}(X) \oplus K_0^{top}(X,\alpha) \oplus \cdots$

We also know that $P(\mathbb{C}) \to X(\mathbb{C})$ is a projectivization of a top. v.b. And,

$$K_0^{top}(P) \cong K_0^{top}(X) \oplus K_0^{top}(X)[\mathcal{O}^{top}(1)] \oplus \cdots$$

Two decompositions coincide. Unwinding spits out $\exp(B) = ch\left(\mathcal{O}^{top}(1)\right)$.

Cor. If $\mathrm{per}(\alpha)$ is prime to $(\dim X - 1)! = (d-1)!$

Then, $\mathrm{ind}_H(\alpha) \mid \mathrm{per}(\alpha)^{d-1}$ for α top. trivial.

If you could prove $IHC(D^b(X,\alpha))$, you'd be done!