Hypercompletion, Postnikov towers and coinductive equivalences

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- Homotopy groups
- Postnikov tower
- t-structures
- Filtrations

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Homotopy theory is largely the systematic study of equivalences, truncation and connectivity.

Homotopy theory provides methods to analyze how far a map of spaces is away from being an equivalence.

Goal: Extend and lift concepts of homotopy theory to higher category theory.

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Extend and lift truncation and connectivity, the Postnikov tower and homotopy groups from topology to higher category theory, where we follow [1].

Extend along embedding

$$\infty\mathrm{Grp} \subset \infty\mathrm{Cat}.$$

Lift along its left adjoint, the groupoidification

$$\infty\mathrm{Cat} \to \infty\mathrm{Grp}.$$

Goal:

Extend the toolkit of algebraic topology to higher category theory to work with higher categories as with spaces. To apply the computational power of algebraic topolgy to higher category theory.

Definition

Let $n \geq -2$. A map of spaces $X \to Y$ is n-connected if the induced map

$$\pi_i(X) \to \pi_i(Y)$$

is a bijection for $i < n+1$ and every choice of base point, and the induced map

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is a surjection.

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A space X is *n*-connected (= map $X \to *$ is *n*-connected).
$\Leftrightarrow$ $\pi_i(X) = 0$ for $i \leq n$ for every choice of base point.

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Let $n \geq -2$. A map of spaces is *n*-truncated if the induced map

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is an injection.

A map is *n*-truncated. $\Leftrightarrow$ All fibers are *n*-truncated $=$ *n*-types.

A space X is *n*-truncated ($X \to *$ is *n*-truncated). $\Leftrightarrow$ X is an *n*-type.

There is a factorization on the category of spaces:

The left class is the class of n-connected maps.

The right class is the class of n-truncated maps.

Let $n \geq -1$. A map of spaces $X \to Y$ is *n*-connected if and only if for every $m \leq n+1$ every commutative square

$$\begin{array}{ccc} S^{m-1} & \longrightarrow & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y \end{array}$$

admits a not necessarily unique filler, where $S^{-1} := \varnothing$.

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The latter formulations extend to category theory:

Remember that S^{m-1} is the groupoidification of $\partial\mathbb{D}^m$ and that $*$ is the groupoidification of $\mathbb{D}^m$.

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Definition

Let $\mathrm{n} \geq -1$. A functor $X \to Y$ is *n*-connected if and only if for every $m \leq n+1$ every commutative square

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By convention, every functor is -2-connected.

Definition

Let $\mathrm{n} \geq -2$. A functor $X \to Y$ is *n*-truncated if and only if for every $m > n+1$ every commutative square

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Definition

Let $n \geq 0$. A functor $X \to Y$ is *n*-full if every commutative square

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Definition

Let $\mathrm{n} \geq 0$. A functor $X \to Y$ is *n*-full if every commutative square

$$\begin{array}{ccc} \partial\mathbb{D}^n & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbb{D}^n & \longrightarrow & Y \end{array}$$

admits a not necessarily unique filler.

A functor is *n*-connected if and only if it is *m*-full for every $m \leq n+1$.

- A functor is -1-connected if and only if it is 0-full, which means essentially surjective.
- A functor is n-connected if and only if it is -1-connected and induces on morphism ∞-categories an $n-1$-connected functor.

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Examples

The functor $\partial\mathbb{D}^n \to \mathbb{D}^n$ is $n-2$-connected.

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- A functor is n-connected if and only if it is -1-connected and induces on morphism ∞-categories an $n-1$-connected functor.

Examples

The functor $\partial\mathbb{D}^n \to \mathbb{D}^n$ is $n-2$-connected.

Examples

For $m \leq n$ the canonical functor $\mathbb{D}^n \to \mathbb{D}^m$ is $m-1$-connected.

A functor is n-truncated if it induces on morphism ∞-categories an $n-1$-truncated functor.

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Example

A functor is -2-truncated if it is an equivalence.

A functor is -1-truncated if it is fully faithful.

A functor is 0-truncated if it is faithful.

Definition

An ∞-category X is n-connected (n-truncated) if it the functor

$$X \to *$$

is n-connected (n-truncated).

Example

An ∞-category X is -1-connected if and only if it the functor $X \to *$ is essentially surjective.

$\implies X$ is -1-connected if and only if X is non-empty.

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An ∞-category X is n-connected if it is -1-connected and all morphism ∞-categories are $n-1$-connected.

Example

An ∞-category X is 0-connected if and only if it is non-empty and for all $A, B \in X$ the morphism ∞-category is non-empty.

$\Leftrightarrow$

X contains an object and for every $A, B \in X$ there is a morphism $A \to B$. But by symmetry there is also a morphism $B \to A$ in X.

Example

An ∞-category X is 1-connected if and only if it is non-empty and for all $A, B \in X$ the morphism ∞-category is 0-connected.

$\Leftrightarrow$

X contains an object, for every objects $A, B \in X$ there are morphisms $f : A \to B$ and $g : B \to A$ in X and for every parallel morphisms $f, g : A \to B$ there are 2-morphisms $f \to g, g \to f$.
But in this case there are also 2-morpisms

$$\mathrm{id} \to gf, gf \to \mathrm{id}, \mathrm{id} \to fg, fg \to \mathrm{id}.$$

Example

Let X be an ∞-category with base point (distinguished object). The reduced suspension

$$\Sigma(X) := S(X)/S(*)$$

increases connectivity by one.

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Example

An ∞-category X is -1-truncated if and only if all morphism ∞-categories are -2-truncated (=contractible).

$\implies$

An ∞-category X is -1-truncated if and only if it is empty or contractible.

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An ∞-category X is -1-truncated if and only if all morphism ∞-categories are -2-truncated (=contractible).

$\Longrightarrow$

An ∞-category X is -1-truncated if and only if it is empty or contractible.

Example

An ∞-category X is 0-truncated if and only if all morphism ∞-categories are -1-truncated (= empty or contractible).

$\Longrightarrow$

An ∞-category X is 0-truncated if and only if it is a poset.

There is is the following inductive terminology:

An $(n, n+1)$-category is a category enriched in the category of $(n-1, n)$-categories.

By convention only the contractible category is a $(-2, -1)$-category.

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Example

A $(-1, 0)$-category is a -1-type $(= \varnothing$ or $*)$.

A $(0, 1)$-category is a category enriched in -1-types $(=$ a poset$)$.

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Corollary

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Proof.

An ∞-category is n-truncated if and only if it all morphism ∞-categories are $n-1$-truncated.

An ∞-category is an $(n, n+1)$-category if and only if it all morphism ∞-categories are $(n-1, n)$-categories.

By induction we reduce to: An ∞-category is -2-truncated if and only if it is contractible $= (-1, -2)$-category.

□

Lemma

Let $n \geq -2$. There is a factorization system on ∞Cat.

The left class is the class of n-connected functors.
The right class is the class of n-truncated functors.

Corollary

There is a localization

$$\tau_{\leq n} : \infty\mathrm{Cat} \to \tau_{\leq n}(\infty\mathrm{Cat})$$

The local objects are the n-truncated ∞-categories = the $(n, n+1)$*-categories.*

For every $X \in \infty\mathrm{Cat}$ *the unit is part of the factorization*

$$X \to \tau_{\leq n}(X) \to *$$

into an n-connected and n-truncated functor.

An ∞-category is n-truncated if and only if it is local with respect to every n-connected functor.

In category the notion of connectivity is more subtle:

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A crucial feature in topology:

a map of spaces is an equivalence if and only if it is n-connected for every $n \geq 0$ since homotopy groups detect equivalences.

$\implies$ A space is contractible if it is n-connected for every $n \geq 0$.

In higher category this does not hold anymore:

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A functor is not an equivalence if it is ∞-connected $:=$ n-connected for every $n \geq 0$ $=$ n-full for every $n \geq 0$.

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In higher category this does not hold anymore:

A functor is not an equivalence if it is ∞-connected := n-connected for every $n \geq 0$ = n-full for every $n \geq 0$.

An ∞-category is not contractible if it is ∞-connected.

The universal functor

$$\mathrm{Bord}_\infty \to \tau_0(\mathrm{Bord}_\infty)$$

to the groupoidification is ∞-connected but not an equivalence.

Bord_∞ is the sequential colimit

$$\mathrm{Bord}_1 \to \mathrm{Bord}_2 \to \dots$$

In the n-category Bord_n every morphism of dimension smaller n admits both adjoints.
$\implies$ In Bord_∞ every morphism admits both adjoints.

Lemma

Let $n \geq 1$. Every n-category X that admits adjoints for morphisms of dimension $k \leq n$, is a space.

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Proof.

By induction on k we can reduce to $k = 1$.
By induction on n we can reduce to $n = 1$.

In every 1-category every morphism that admits a right adjoint is an equivalence since unit and counit are equivalences.

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By induction on k we can reduce to $k = 1$.
By induction on n we can reduce to $n = 1$.

In every 1-category every morphism that admits a right adjoint is an equivalence since unit and counit are equivalences. □

Let X be an ∞-category that admits adjoints of dimension $\leq n+1$. The functor $X \to \tau_{\leq n}(X)$ is n-connected. $\implies$ $\tau_{\leq n}(X)$ admits adjoints of dimension $\leq n+1$ and so is the space $\tau_{\leq 0}(X)$.

$\implies$ for every $n \geq 0$ the functor $\mathrm{Bord}_\infty \to \tau_{\leq 0}(\mathrm{Bord}_\infty)$ is equivalent to the n-connected functor

$$\mathrm{Bord}_\infty \to \tau_{\leq n}(\mathrm{Bord}_\infty).$$

$\implies$ The functor $\mathrm{Bord}_\infty \to \tau_{\leq 0}(\mathrm{Bord}_\infty)$ is ∞-connected.

There is a remedy:

Definition

1. A functor is an 0-equivalence if and only if it induces a bijection on the set of equivalences classes of objects.
2. A functor is an n-equivalence if and only if it induces $n-1$-equivalences on morphism ∞-categorires.

A functor $X \to Y$ is an n-equivalence if and only if it induces an equivalence

$$Ho_n(X) \to Ho_n(Y).$$

(and so also on Ho_m for $m < n$)).

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For n-equivalences the relationship extends from topology:

A functor is an equivalence if it is an n-equivalence for every $n \geq 0$.

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However, n-equivalences do not formally behave that well:

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For n-equivalences the relationship extends from topology:

A functor is an equivalence if it is an n-equivalence for every $n \geq 0$.

However, n-equivalences do not formally behave that well:

The class of n-equivalences is not the left class of a factorization system. It is the right class of a factorization system. The left class is generated by the inclusions $\partial\mathbb{D}^m \subset \mathbb{D}^m$ for $m \leq n$:

For instance: the functor $\varnothing \to X$ factors as $\varnothing \to \iota_n(X) \to X$.

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Definition

Let X be an ∞-category.
A 0-connection in X is a collection $(A, B, f : A \to B, g : B \to A)$ of objects $A, B \in X$ and 1-morphisms f, g in X.

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We say that $(A, B, f : A \to B, g : B \to A)$ is a 0-connection between A and B. If a 0-connection exists between A and B in X, we say that A and B are 0-connected.

Definition

An $n+1$-connection in X is a collection $(A, B, f, g, \sigma, \epsilon)$, where $(A, B, f : A \to B, g : B \to A)$ is a 0-connection and σ is an n-connection between id_A and gf in $\mathrm{Mor}_X(A, B)$ and ϵ is an n-connection between id_B and fg in $\mathrm{Mor}_X(B, A)$.

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We say that $(A, B, f, g, \sigma, \epsilon)$ is an n-connection between A and B. If an n-connection exists between A and B in X, we say that A and B are n-connected.

Example

A 1-connection is a collection $(A, B, f, g, \alpha, \beta, \gamma, \delta)$, where $A, B \in X$ and $f : A \to B, g : B \to A$ morphisms in X and

$$\alpha : \mathrm{id} \to gf, \beta : gf \to \mathrm{id}, \gamma : \mathrm{id} \to fg, \delta : fg \to \mathrm{id}$$

are 2-morphisms in X.

Example

A 2-connection is a collection $(A, B, f, g, \alpha, \beta, \gamma, \delta)$, where $A, B \in X$ and $f : A \to B, g : B \to A$ morphisms in X and

$$\alpha : \mathrm{id} \to gf, \beta : gf \to \mathrm{id}, \gamma : \mathrm{id} \to fg, \delta : fg \to \mathrm{id}$$

are 2-morphisms in X and 3-morphisms

$$\mathrm{id} \to \beta\alpha, \beta\alpha \to \mathrm{id}, \mathrm{id} \to \alpha\beta, \alpha\beta \to \mathrm{id}, \mathrm{id} \to \delta\gamma, \delta\gamma \to \mathrm{id}, \mathrm{id} \to \gamma\delta, \gamma\delta \to \mathrm{id}.$$

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Example

An ∞-category is 1-connected if and only if there is a 1-connection between any two objects and a 0-connection between any two parallel morphisms.

Definition

An ∞-connection is a system of n-connections for $n \geq 0$ compatibly extending each other.

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Let D^2 be the 2-category classifying a pair of objects A, B, morphisms $f : A \to B, g : B \to A$ and 2-morphims $\mathrm{id} \to gf, fg \to \mathrm{id}, \mathrm{id} \to fg, gf \to \mathrm{id}$.

There is a canonical functor $\mathbb{D}^1 \to D^2$ taking the generator of f.

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There is a canonical functor $\mathbb{D}^1 \to D^2$ taking the generator of f.

Let D^{n+1} be the pushout

$$\begin{array}{ccc} \coprod_{4^{n-1}} S^{n-1}(\mathbb{D}^1) & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ \coprod_{4^{n-1}} S^{n-1}(D^2) & \longrightarrow & D^{n+1} \end{array}$$

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D^{n+1} is n-connected for every $n \geq 2$ and so D^∞ is ∞-connected.

Let $1 \leq n \leq \infty$ and X an ∞-category. An n-connection in X is a functor $D^{n+1} \to X$.

Definition

Two objects of an ∞-category X are ∞-connected if there is an ∞-connection between these objects.

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Lemma

An ∞-category is ∞-connected if and only if any two parallel m-morphisms for $m \geq 0$ are ∞-connected.

Definition

Two objects of an ∞-category X are ∞-connected if there is an ∞-connection between these objects.

Lemma

An ∞-category is ∞-connected if and only if any two parallel m-morphisms for $m \geq 0$ are ∞-connected.

An ∞-category X is ∞-truncated (:= local with respect to every ∞-connected functor) if and only if every ∞-connection in X and in all iterated morphism ∞-categories of X is an equivalence.

An ∞-connected functor from an ∞-truncated ∞-category is an equivalence.
$\Leftrightarrow$ An ∞-connected functor $\phi : X \to Y$ starting at a ∞-truncated ∞-category is a n-equivalence for every $n \geq 0$, which follows by induction:

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$\Leftrightarrow$ An ∞-connected functor $\phi : X \to Y$ starting at a ∞-truncated ∞-category is a n-equivalence for every $n \geq 0$, which follows by induction:

Induction start: $\phi : X \to Y$ is an 0-equivalence. We know that ϕ is essentially surjective. We have to see that ϕ induces an injection on equivalence classes of objects:

Let $A, B \in X$ and $\sigma : \phi(A) \simeq \phi(B)$. Then σ determines an ∞-connection between $\phi(A), \phi(B)$ that lifts along ϕ to an ∞-connection between A, B since ϕ is ∞-connected.

$\implies A \simeq B$ since X is ∞-truncated.

Definition

Let X be an ∞-category. The poset of components of X, denoted by $\pi_0(X)$, is the quotient of the set of equivalence classes of objects of X with respect to the equivalence relation

$$A \sim B \Leftrightarrow \text{there is a 0-connection between } A \text{ and } B.$$

We set

$$A \leq B \Leftrightarrow \text{there is a morphism } f : A \to B.$$

We make the following inductive definition:

Definition

1. A functor $\phi : X \to Y$ is a Postnikov 0-equivalence if it induces an equivalence on components.
2. A functor $\phi : X \to Y$ is a Postnikov n-equivalence if it induces Postnikov $n-1$-equivalences on morphism ∞-categories and for every $t \in Y$ there is an $s \in X$ and an n-connection between $\phi(s)$ and t.

Lemma

Every n-connected functor is a Postnikov n-equivalence.

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Every n-connected functor is a Postnikov n-equivalence.

By induction on $n \geq 0$ reduce to $n = 0$.
A 0-connected functor $\phi : X \to Y$ is essentially surjective and full.
$\implies$ the functor $\pi_0(X) \to \pi_0(Y)$ is surjective. It is also injective:
If $A, B \in X$ have the same image in $\pi_0(Y)$, there are morphisms $\phi(A) \to \phi(B)$ and $\phi(B) \to \phi(A)$ in Y. By fullness there are morphisms $A \to B$ and $B \to A$ in X.

Categorical Hurewicz theorem [2, Theorem 5.42.]:

Theorem

Let X be an ∞-category with distinguished object that is the n-fold delooping of an $\mathbb{E}_n$-monoidal ∞-category.

The functor $X \simeq S^0 \wedge X \to H(\mathbb{N}) \wedge X$ is a Postnikov n-equivalence.

Let X be an ∞-category. Then $\pi_0(X) = \tau_{\leq 0}(X)$.

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Let X be an ∞-category.

Then $\tau_{\leq n}(X)$ is the "quotient" of $\mathrm{Ho}_n(X)$ by identifying objects up to n-connection, morphisms up to $n-1$-connection, 2-morphisms up to $n-2$-connection, etc.

Definition

Let X be an ∞-category. The Postnikov tower of X is the tower

$$... \to \tau_{\leq 2}(X) \to \tau_{\leq 1}(X) \to \tau_{\leq 0}(X)$$

The functor $\tau_{\leq n+1}(X) \to \tau_{\leq n}(X)$ is n-connected.

Definition

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The functor $\tau_{\leq n+1}(X) \to \tau_{\leq n}(X)$ is n-connected.

Definition

Let X be an ∞-category. The Postnikov completion of X is the functor

$$X \to \tau_{\leq \infty}(X) := \lim(\ldots \to \tau_{\leq 2}(X) \to \tau_{\leq 1}(X) \to \tau_{\leq 0}(X))$$

Example

Let X be a space. Then $\tau_{\leq n}(X)$ is a space since $X \to \tau_{\leq n}(X)$ is n-connected so that every morphism of $\tau_{\leq n}(X)$ lifts to X.
$\implies$ The Postnikov tower of a space is a tower of spaces, which is the classical Postnikov tower.

Definition

For every ∞-category X the Postnikov completion

$$\tau_{\leq\infty}(X) = \lim_{n\geq 0} \tau_{\leq n}(X)$$

is ∞-truncated (local with respect to every ∞-connected functor).

Proof.

Every ∞-connected functor $f : A \to B$ is n-connected for every $n \geq 0$.

$\implies$ The map

$$\lim_{n\geq 0} \mathrm{Fun}(B, \tau_{\leq n}(X)) \simeq \mathrm{Fun}(B, \lim_{n\geq 0} \tau_{\leq n}(X)) \to$$

$$\mathrm{Fun}(A, \lim_{n\geq 0} \tau_{\leq n}(X)) \simeq \lim_{n\geq 0} \mathrm{Fun}(A, \tau_{\leq n}(X))$$

is an equivalence since $\tau_{\leq n}(X)$ is n-truncated and so local with respect to every n-connected functor.

□

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An ∞-category X is directed if any two objects are equivalent if there is a morphism $A \to B$ and $B \to A$, and the same for iterated morphism ∞-categories.

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Example

Every Steiner ∞-category and more generally loopfree gaunt ∞-category is directed. Every space is directed.

Directedness = "loopfreeness without gauntness."

Let X be a directed ∞-category. Then $\tau_{\leq 0}(X)$ is the set of equivalence classes of objects of X. More generally, the functor

$$\mathrm{Ho}_n(X) \to \tau_{\leq n}(X)$$

is an equivalence of underlying n-categories.
$\implies$ The Postnikov tower of X is a tower of homotopy categories

$$... \to \mathrm{Ho}_2(X) \to \mathrm{Ho}_1(X) \to \mathrm{Ho}_0(X).$$

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Let X be an ∞-category. In general, in the Postnikov tower of X the set of equivalence classes of objects is changing from one level to the next: for descending n objects get identified more and more via the equivalence relation of being n-connected.

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This makes it hard for the Postnikov completion $X \to \lim_{n \geq 0} \tau_{\leq n}(X)$ to be essentially surjective.
So in general not every ∞-category is Postnikov complete.

Spaces are Postnikov complete. By induction one can prove that (∞, n)-categories are Postnikov complete:
For any $A, B \in X$ the functor

$$X \to \tau_{\leq\infty}(X) = \lim_{n\geq 0} \tau_{\leq n}(X)$$

induces on morphism ∞-categories the functor

$$\mathrm{Mor}_X(A, B) \to \tau_{\leq\infty}(\mathrm{Mor}_X(A, B)).$$

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Directed ∞-categories are Postnikov complete. In particular, loopfree ∞-categories are Postnikov complete.

David Gepner and Hadrian Heine.
Homotopy posets, Postnikov towers, and hypercompletions of ∞-categories.
arXiv:2603.09903, 2026.

Hadrian Heine.
On the categorification of homology.
arXiv:2505.22640, 2025.