Homotopy posets and the oriented exact sequence

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Lemma

Let $n \geq -2$. There is a factorization system on ∞Cat.

The left class is the class of n-connected functors.
The right class is the class of n-truncated functors.

Corollary

There is a localization

$$\tau_{\leq n} : \infty\mathrm{Cat} \to \tau_{\leq n}(\infty\mathrm{Cat})$$

The local objects are the n-truncated ∞-categories = the $(n, n+1)$-categories.

For every $X \in \infty\mathrm{Cat}$ the unit is part of the factorization

$$X \to \tau_{\leq n}(X) \to *$$

into an n-connected and n-truncated functor.

Definition

Let X be an ∞-category. The Postnikov tower of X is the tower

$$... \to \tau_{\leq 2}(X) \to \tau_{\leq 1}(X) \to \tau_{\leq 0}(X)$$

The functor $\tau_{\leq n+1}(X) \to \tau_{\leq n}(X)$ is n-connected.

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The $(n, n+1)$-category $\tau_{\leq n}(X)$ is the "quotient" of $\mathrm{Ho}_n(X)$ by identifying objects up to n-connection, morphisms up to $n-1$-connection, 2-morphisms up to $n-2$-connection, etc.

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Definition

Let X be an ∞-category. The Postnikov completion of X is the functor

$$X \to \tau_{\leq \infty}(X) := \lim(... \to \tau_{\leq 2}(X) \to \tau_{\leq 1}(X) \to \tau_{\leq 0}(X))$$

Example

Let X be a space. The Postnikov tower of X is the classical Postnikov tower.

Definition

An ∞-category X is directed if any two objects are equivalent if there is a morphism $A \to B$ and $B \to A$, and the same for iterated morphism ∞-categories.

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Example

Every Steiner ∞-category and more generally loopfree gaunt ∞-category is directed. Every space is directed.

Directedness = "loopfreeness without gauntness."

Let X be a directed ∞-category. Then the Postnikov tower of X is a tower of homotopy categories

$$... \to \mathrm{Ho}_2(X) \to \mathrm{Ho}_1(X) \to \mathrm{Ho}_0(X).$$

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Directed ∞-categories are Postnikov complete. In particular, loopfree ∞-categories are Postnikov complete.

We defined $\infty\mathrm{Cat}$ as the limit of the tower

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There is a canonical equivalence

$$\lim_{n\geq 0} n\mathrm{Cat} \simeq \lim_{n\geq 0} \lim_{m\geq n-1} (m,n)\mathrm{Cat} \simeq \lim_{n\geq 0} (n-1,n)\mathrm{Cat} \simeq$$

$$\lim_{n\geq 0} (n,n+1)\mathrm{Cat} = \lim_{n\geq 0} \tau_{\leq n}\infty\mathrm{Cat}$$

using the left adjoints of the embeddings and $\tau_{\leq n}$.

[1, Theorem 3.6.15.]:

Theorem

Let $n \geq 0$. *The functor*

$$\tau_{\leq\infty} : \infty\mathrm{Cat} \to \lim_{n\geq 0} \tau_{\leq n}\infty\mathrm{Cat} = \lim_{n\geq 0}(n, n+1)\mathrm{Cat}$$

admits a fully faithful right adjoint R.
The unit of the adjunction at $X \in \infty\mathrm{Cat}$ *is the Postnikov completion*

$$X \to R(\tau_{\leq\infty}(X)) \simeq \lim_{n\geq 0} \tau_{\leq n}(X).$$

Hence an ∞*-category* X *lies in the essential image of* R *if and only if it is Postnikov complete.*

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We have a similar relationship by replacing fibers by oriented fibers.

Definition

Let $A \to C, B \to C$ be functors.

1. The oriented pullback of $A \to C, B \to C$ is

$$A \underset{C}{\vec{\times}} B := A \underset{C}{\times} \mathrm{Fun}^{\mathrm{oplax}}(\mathbb{D}^1, C) \underset{C}{\times} B.$$

2. The antioriented pullback of $A \to C, B \to C$ is

$$A \underset{C}{\overline{\vec{\times}}} B := A \underset{C}{\times} \mathrm{Fun}^{\mathrm{lax}}(\mathbb{D}^1, C) \underset{C}{\times} B.$$

Example

Let C be an ∞-category.
Then
$$C \underset{C}{\overset{\rightarrow}{\times}} C := \mathrm{Fun}^{\mathrm{oplax}}(\mathbb{D}^1, C).$$

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Example

Let C be an ∞-category and $X \in C$.
Then

$$\{X\} \underset{C}{\vec{\times}} C = \mathcal{C}_{X//} := \{X\} \times_{C^{\{0\}}} \mathrm{Fun}^{\mathrm{oplax}}(\mathbb{D}^1, C).$$

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$$C \underset{C}{\vec{\times}} \{X\} = \mathcal{C}_{//X} := \{X\} \times_{C^{\{1\}}} \mathrm{Fun}^{\mathrm{oplax}}(\mathbb{D}^1, C).$$

Example

Let C be an ∞-category and $X, Y \in C$.
There is an equivalence

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Let $A \to C, B \to C$ be functors.

1. There is an equivalence:

$$(A \underset{C}{\vec{\times}} B)^{\mathrm{co}} \simeq A^{\mathrm{co}} \underset{C^{\mathrm{co}}}{\overline{\vec{\times}}} B^{\mathrm{co}}.$$

2. There is an equivalence:

$$(A \underset{C}{\vec{\times}} B)^{\mathrm{op}} \simeq B^{\mathrm{op}} \underset{C^{\mathrm{op}}}{\overline{\vec{\times}}} A^{\mathrm{op}}.$$

The following is [1, Theorem 3.3.32.]:

Theorem

Let $n \geq -2$. Let $X \to S$, $Y \to S$ be functors and $\phi : X \to Y$ a functor over S. The following are equivalent:

1. *The functor $\phi : X \to Y$ is n-connected (n-truncated).*
2. *For every functor $T \to S$ the induced functor $T \overset{\to}{\times}_S \phi : T \overset{\to}{\times}_S X \to T \overset{\to}{\times}_S Y$ is n-connected (n-truncated).*
3. *For every object $s \in S$ the induced functor $\{s\} \overset{\to}{\times}_S \phi : \{s\} \overset{\to}{\times}_S X \to \{s\} \overset{\to}{\times}_S Y$ is n-connected (n-truncated).*

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A map of spaces $X \to Y$ is n-connected if and only if it is 0-connected and the diagonal map

$$X \to X \times_Y X$$

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In topology we have an inductive definition of n-connected and n-truncated maps via iterated pullbacks:

A map of spaces $X \to Y$ is n-connected if and only if it is 0-connected and the diagonal map

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is $n-1$-connected.

A map of spaces $X \to Y$ is n-truncated if and only if the diagonal map

$$X \to X \times_Y X$$

is $n-1$-truncated.

The following is [1, Proposition 3.4.25.]:

Theorem

Let $n \geq -1$.

1. *A functor $X \to Y$ is n-truncated if and only if the canonical functor*
$$X \vec{\times}_X X \to X \vec{\times}_Y X$$
is $n-1$-truncated. A functor is -2-truncated if and only if it is an equivalence.
2. *A functor $X \to Y$ is n-connected if and only if it is essentially surjective and the canonical functor*
$$X \vec{\times}_X X \to X \vec{\times}_Y X$$
is $n-1$-connected. Every functor is -2-connected.

Proof.

Let $\phi : X \to Y$ be a functor. The functor

$$X \overset{\to}{\times}_X X \to X \overset{\to}{\times}_Y X$$

over $X \times X$ induces on the fiber over every $(A, B) \in X \times X$ the induced functor $\mathrm{Mor}_X(A, B) \to \mathrm{Mor}_Y(\phi(A), \phi(B))$.

□

A useful features of oriented pullbacks is that we can compute morphism ∞-categories of them:

Lemma

Let $F : \mathcal{A} \to \mathcal{C}, G : \mathcal{B} \to \mathcal{C}$ be functors and $A, A' \in \mathcal{A}, B, B' \in \mathcal{B}$ and $\sigma : F(\mathrm{A}) \to G(\mathrm{B}), \sigma' : F(\mathrm{A}') \to G(\mathrm{B}')$ morphisms.
There is an equivalence

$$\mathrm{Mor}_{\mathcal{A} \overset{\to}{\underset{\mathcal{C}}{\times}} \mathcal{B}}((\mathrm{A}, \mathrm{B}, \sigma), (\mathrm{A}', \mathrm{B}', \sigma')) \simeq$$

$$\mathrm{Mor}_{\mathcal{B}}(\mathrm{B}, \mathrm{B}') \underset{\mathrm{Mor}_{\mathcal{C}}(F(\mathrm{A}), G(\mathrm{B}'))}{\overset{\to}{\times}} \mathrm{Mor}_{\mathcal{A}}(\mathrm{A}, \mathrm{A}').$$

The following is [1, Proposition 3.5.1.]:

Theorem

Let $n \geq -2$ and

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ \mathcal{A}' & \xrightarrow{F'} & \mathcal{C}', \end{array} \quad \begin{array}{ccc} \mathcal{B} & \xrightarrow{G} & \mathcal{C} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ \mathcal{B}' & \xrightarrow{G'} & \mathcal{C}' \end{array}$$

commutative squares of ∞-categories. If α, β, γ are n-truncated, the induced functor

$$\mathcal{A} \overset{\rightarrow}{\times}_{\mathcal{C}} \mathcal{B} \to \mathcal{A}' \overset{\rightarrow}{\times}_{\mathcal{C}'} \mathcal{B}'$$

is n-truncated.

Proof.

The induced functor induces on morphism ∞-categories the functor

$$\mathrm{Mor}_{\mathcal{A} \underset{\mathcal{C}}{\vec{\times}} \mathcal{B}}((A, B, \sigma), (X, Y, \rho)) \to \mathrm{Mor}_{\mathcal{A}' \underset{\mathcal{C}'}{\vec{\times}} \mathcal{B}'}((A', B', \sigma'), (X', Y', \rho')).$$

It identifies with the functor

$$\mathrm{Mor}_{\mathcal{B}}(B, Y) \underset{\mathrm{Mor}_{\mathcal{C}}(F(A), G(Y))}{\vec{\times}} \mathrm{Mor}_{\mathcal{A}}(A, X) \to$$

$$\mathrm{Mor}_{\mathcal{B}'}(B', Y') \underset{\mathrm{Mor}_{\mathcal{C}'}(F'(A'), G'(Y'))}{\vec{\times}} \mathrm{Mor}_{\mathcal{A}'}(A', X').$$

We apply induction on $n \geq -2$.

□

Corollary

Let $n \geq -2$ and $\phi : \mathcal{A} \to \mathcal{B}$ an n-truncated functor and $\mathcal{C}$ an ∞-category. The induced functors

$$\mathrm{Fun}^{\mathrm{oplax}}(\mathcal{C}, \mathcal{A}) \to \mathrm{Fun}^{\mathrm{oplax}}(\mathcal{C}, \mathcal{B}),$$

$$\mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{A}) \to \mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{B}),$$

are n-truncated.

Corollary

Let $n \geq -2$. The Gray tensor product of two n-connected functors is again n-connected.

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Proof.

Let $f : A \to B$ be an n-connected functor and C an ∞-category. The functor $f \boxtimes C : A \boxtimes C \to B \boxtimes C$ is n-connected if and only if it has the left lifting property with respect to every n-truncated functor $X \to Y$.

By adjointness f is n-connected if and only if f has the left lifting property with respect to the functor

$$\mathrm{Fun}^{\mathrm{oplax}}(C, X) \to \mathrm{Fun}^{\mathrm{oplax}}(C, Y),$$

which is n-truncated.
Similarly for $C \boxtimes f$ and $\mathrm{Fun}^{\mathrm{lax}}(-,-)$.

Corollary

The localization $\tau_{\leq n} : \infty\mathrm{Cat} \to \tau_{\leq n}\infty\mathrm{Cat} = (n, n+1)\mathrm{Cat}$ *is a monoidal localization for the Gray tensor product.*

$\implies$ *The functor* $\tau_{\leq n} : \infty\mathrm{Cat} \to \tau_{\leq n}\infty\mathrm{Cat} \subset \infty\mathrm{Cat}$ *is lax monoidal for the Gray tensor product.*

The following is [1, Theorem 3.3.32.]:

Theorem

Let $\mathrm{n} \geq -2$ *and*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ \mathcal{A}' & \xrightarrow{F'} & \mathcal{C}', \end{array} \qquad \begin{array}{ccc} \mathcal{B} & \xrightarrow{G} & \mathcal{C} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ \mathcal{B}' & \xrightarrow{G'} & \mathcal{C}' \end{array}$$

commutative squares of ∞*-categories. If* α, β *are n-connected and* γ *is* $n+1$*-connected, the induced functor*

$$\mathcal{A} \overset{\rightarrow}{\times}_{\mathcal{C}} \mathcal{B} \to \mathcal{A}' \overset{\rightarrow}{\times}_{\mathcal{C}'} \mathcal{B}'$$

is n*-connected.*

Induction on $n \geq -2$.

The induction start $n = -2$ is trivial: every functor is -2-truncated.

The induction step follows from the formula for morphism ∞-categories in oriented pullbacks, and the statement for $n = 0$.

It suffices to show the statement for $n = 0$: the induced functor

$$\mathcal{A} \overset{\rightarrow}{\times}_{\mathcal{C}} \mathcal{B} \to \mathcal{A}' \overset{\rightarrow}{\times}_{\mathcal{C}'} \mathcal{B}'$$

is essentially surjective if α, β are essentially surjective and γ is full and essentially surjective.
Let

$$(A', B', \sigma' : F'(A') \to G'(B')) \in \mathcal{A}' \overset{\rightarrow}{\times}_{\mathcal{C}'} \mathcal{B}',$$

Then there are $A \in \mathcal{A}, B \in \mathcal{B}$ such that $A' \simeq \alpha(A), B' \simeq \beta(B)$.
Since γ is full, the morphism

$$\gamma(F(A)) \simeq F'(\alpha(A)) \simeq F'(A') \xrightarrow{\sigma'} G'(B') \simeq G'(\beta(B)) \simeq \gamma(G(B))$$

in $\mathcal{C}'$ is the image of a morphism $\sigma : F(A) \to G(B)$ in $\mathcal{C}$.
Then $(A, B, \sigma) \in \mathcal{A} \overset{\rightarrow}{\times}_{\mathcal{C}} \mathcal{B}$ lies over (A', B', σ').

Corollary

Let $\alpha : \mathcal{A} \to \mathcal{C}, \beta : \mathcal{B} \to \mathcal{C}$ *be functors and* $\mathrm{n} \geq -1$. *The canonical functor*

$$\mathcal{A} \overset{\to}{\times}_{\mathcal{C}} \mathcal{B} \to \tau_{\leq n}(\mathcal{A}) \overset{\to}{\times}_{\tau_{\leq n}(\mathcal{C})} \tau_{\leq n}(\mathcal{B})$$

is $\mathrm{n}-1$*-connected.*

$\implies$ *The canonical functor*

$$\tau_{\leq n}(\mathcal{A} \overset{\to}{\times}_{\mathcal{C}} \mathcal{B}) \to \tau_{\leq n}(\mathcal{A}) \overset{\to}{\times}_{\tau_{\leq n}(\mathcal{C})} \tau_{\leq n}(\mathcal{B})$$

is $\mathrm{n}-1$*-connected.*

Definition

Let $\mathcal{C}$ be an ∞-category and $n \geq 0$.

1. An oriented base point of $\mathcal{C}$ of dimension n is a sequence of pairs
$$(X_i, Y_i)_{0 \leq i \leq n}$$
such that (X_i, Y_i) are i-morphisms $X_{i-1} \to Y_{i-1}$ in $\mathcal{C}$ for every $0 \leq i \leq n$.
2. An oriented base point of $\mathcal{C}$ is a sequence of pairs
$$(X_i, Y_i)_{i \geq 0}$$
such that (X_i, Y_i) are i-morphisms $X_{i-1} \to Y_{i-1}$ in $\mathcal{C}$ for every $i \geq 0$.

The inclusion $\varnothing \to \partial\mathbb{D}^1$ induces an inclusion

$$\partial\mathbb{D}^{\mathrm{n}} = S^n(\varnothing) \to S^n(\partial\mathbb{D}^1) = S^{\mathrm{n+1}}(\varnothing) = \partial\mathbb{D}^{\mathrm{n+1}}.$$

Let

$$\partial\mathbb{D}^\infty := \mathrm{colim}(\partial\mathbb{D}^0 \to \ldots \to \partial\mathbb{D}^n \to \ldots)$$

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Let

$$\partial\mathbb{D}^\infty := \mathrm{colim}(\partial\mathbb{D}^0 \to \ldots \to \partial\mathbb{D}^n \to \ldots)$$

Let X be an ∞-category and $\mathrm{n} \geq 0$. An oriented base point of X of dimension n is precisely a functor $\partial\mathbb{D}^{\mathrm{n}+1} \to X$. An oriented base point of X is precisely a functor $\partial\mathbb{D}^\infty \to X$.

Definition

Let X be an ∞-category.

1. We set
$$\mathrm{Mor}_X^0(Z) := X.$$
2. Let $\mathrm{n} \geq 1$ and $Z := (X_{\mathrm{i}}, Y_{\mathrm{i}})_{0 \leq \mathrm{i} \leq n-1}$ an oriented base point of X of dimension $\mathrm{n} - 1$. We set
$$\mathrm{Mor}_X^{\mathrm{n}}(Z) := \mathrm{Mor}_{\mathrm{Mor}_X^{\mathrm{n}-1}(Z)}(X_{\mathrm{n}-1}, Y_{\mathrm{n}-1}).$$

Lemma

Let X be an ∞-category and $\mathrm{n} \geq 0$. Let $Z := (X_{\mathrm{i}}, Y_{\mathrm{i}})_{0 \leq \mathrm{i} \leq n-1}$ be an oriented base point of X of dimension $\mathrm{n} - 1$ corrresponding to a functor $\partial \mathbb{D}^{\mathrm{n}} \to X$. There is a canonical equivalence

$$\mathrm{Mor}^{\mathrm{n}}_X(Z) \simeq \mathrm{Fun}^{\mathrm{oplax}}_{\partial \mathbb{D}^{\mathrm{n}}/}(\mathbb{D}^{\mathrm{n}}, X).$$

Definition

Let X be an ∞-category. The poset of components of X is the poset

$$\pi_0(X) := \tau_{\leq 0}(X).$$

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Definition

Let $n \geq 1$ and X an ∞-category and Z an oriented base point of X of dimension $n-1$.
The n-th homotopy poset of X is

$$\pi_{\mathrm{n}}(X, Z) := \pi_0(\mathrm{Mor}^{\mathrm{n}}_X(Z)).$$

Lemma

Let X be an ∞-category and $\mathrm{n} \geq 0$. Let $Z := (X_{\mathrm{i}}, Y_{\mathrm{i}})_{0 \leq \mathrm{i} \leq n}$ be an oriented base point of X of dimension n. The oriented base point

$$\bar{Z} := (X_1, \mathrm{id}_{Y_0}, \mathrm{id}_{X_1}, Y_2, X_3, \mathrm{id}_{Y_2}, \mathrm{id}_{X_3}, Y_4, \ldots)$$

of $X_{//Y_0}$ is sent by the forgetful functor $X_{//Y_0} \to X$ to the oriented base point $(X_{\mathrm{i}}, Y_{\mathrm{i}})_{\mathrm{i} \geq 0}$ of X.

Corollary

Let $\phi : X \to Y$ *be a functor and* $\mathrm{n} \geq 0$*. Let* $Z := (X_{\mathrm{i}}, Y_{\mathrm{i}})_{0 \leq \mathrm{i} \leq n}$ *be an oriented base point of* X *of dimension* n*. Then*

$$(Z, \overline{\phi Z})$$

is an oriented base point of dimension n *of the oriented right fiber*

$$X \overset{\rightarrow}{\times}_Y \{\phi(Y_0)\} \simeq X \times_Y Y_{//\phi(Y_0)}.$$

Let X be an ∞-category and $n \geq 0$. Let $Z := (X_{\mathrm{i}}, Y_{\mathrm{i}})_{0 \leq \mathrm{i} \leq n}$ and $Z' := (X'_{\mathrm{i}}, Y'_{\mathrm{i}})_{0 \leq \mathrm{i} \leq n}$ oriented base points of X of dimension n.

By induction on $n \geq 0$ we define what a morphism of oriented base points $Z \to Z'$ of X of dimension n is:

Let X be an ∞-category and $n \geq 0$. Let $Z := (X_{\mathrm{i}}, Y_{\mathrm{i}})_{0 \leq \mathrm{i} \leq n}$ and $Z' := (X_{\mathrm{i}}', Y_{\mathrm{i}}')_{0 \leq \mathrm{i} \leq n}$ oriented base points of X of dimension n.

By induction on $n \geq 0$ we define what a morphism of oriented base points $Z \to Z'$ of X of dimension n is:

Definition

For $n = 0$ a morphism of oriented base points $Z \to Z'$ of X of dimension 0 is a pair of morphisms $X_0' \to X_0$, $Y_0 \to Y_0'$.

Let $Z := (X_{\mathrm{i}}, Y_{\mathrm{i}})_{0 \leq \mathrm{i} \leq n}$ and $Z' := (X'_{\mathrm{i}}, Y'_{\mathrm{i}})_{0 \leq \mathrm{i} \leq n}$ be oriented base points of $\mathcal{C}$ of dimension n.
These correspond to objects X_0, Y_0, X'_0, Y'_0 and oriented base points $\widetilde{Z}$ of $\mathrm{Mor}_X(X_0, Y_0)$ and $\widetilde{Z}'$ of $\mathrm{Mor}_X(X'_0, Y'_0)$ of dimension $n - 1$.

Let $Z := (X_{\mathrm{i}}, Y_{\mathrm{i}})_{0 \leq \mathrm{i} \leq n}$ and $Z' := (X'_{\mathrm{i}}, Y'_{\mathrm{i}})_{0 \leq \mathrm{i} \leq n}$ be oriented base points of $\mathcal{C}$ of dimension n.
These correspond to objects X_0, Y_0, X'_0, Y'_0 and oriented base points $\widetilde{Z}$ of $\mathrm{Mor}_X(X_0, Y_0)$ and $\widetilde{Z}'$ of $\mathrm{Mor}_X(X'_0, Y'_0)$ of dimension $n-1$.

Definition

A morphism of oriented base points $Z \to Z'$ of X of dimension n is the following:

- A pair of morphisms $X'_0 \to X_0, Y_0 \to Y'_0$.
  These give rise to a functor

$$\theta : \mathrm{Mor}_X(X_0, Y_0) \to \mathrm{Mor}_X(X'_0, Y'_0).$$

- A morphism of oriented base points

$$\theta(\widetilde{Z}) \to \widetilde{Z}'$$

  of $\mathrm{Mor}_X(X'_0, Y'_0)$ of dimension $n-1$.

Lemma

Let $n \geq 0$ *and* $\phi : X \to Y$ *a functor and* $Z := (X_\mathrm{i}, Y_\mathrm{i})_{0 \leq \mathrm{i} \leq n}$ *an oriented base point of* X *of dimension* $n-1$. *Let* ν *be the functor*

$$\mathrm{Mor}_Y(\phi(X_0), \phi(Y_0)) \to X \times_Y Y_{//\phi(Y_0)}.$$

There is a canonical morphism

$$(\mathrm{id}_{X_1}, Y_1, X_2, \mathrm{id}_{Y_2}, \mathrm{id}_{X_3}, Y_3, ...) : \nu\phi Z \to (Z, \overline{\phi Z})$$

of oriented base points of dimension n *in* $X \times_Y Y_{//\phi(Y_0)}$.

Let $\phi : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, $Z := (X_i, Y_i)_{i \geq 0}$ an oriented base point of $\mathcal{C}$. Let $\mathcal{F}$ be the oriented right fiber of ϕ over Y_0. There is a long sequence of ∞-categories

$$... \xrightarrow{\delta_2} \mathrm{Mor}^2_{\mathcal{F}}(\bar{Z}) \to \mathrm{Mor}^2_{\mathcal{C}}(Z) \to \mathrm{Mor}^2_{\mathcal{D}}(\phi Z)$$

$$\xrightarrow{\delta_1} \mathrm{Mor}^1_{\mathcal{F}}(\bar{Z}) \to \mathrm{Mor}^1_{\mathcal{C}}(Z) \to \mathrm{Mor}^1_{\mathcal{D}}(\phi Z)$$

$$\xrightarrow{\delta_0} \mathrm{Mor}^0_{\mathcal{F}}(\bar{Z}) \to \mathrm{Mor}^0_{\mathcal{C}}(Z) \to \mathrm{Mor}^0_{\mathcal{D}}(\phi Z)$$

The short sequence

$$\mathrm{Mor}^i_{\mathcal{F}}(\bar{Z}) \to \mathrm{Mor}^i_{\mathcal{C}}(Z) \to \mathrm{Mor}^i_{\mathcal{D}}(\phi Z)$$

is an oriented fiber sequence if i is even, and an antioriented fiber sequence if i is odd. The next short sequence is a fiber sequence:

$$\mathrm{Mor}^{i+1}_{\mathcal{D}}(\phi Z) \xrightarrow{\delta_i} \mathrm{Mor}^i_{\mathcal{F}}(\bar{Z}) \to \mathrm{Mor}^i_{\mathcal{C}}(Z)$$

The functor $\pi_0 : \infty\mathrm{Cat} \to \mathrm{Poset}$ is an oriented functor and therefore sends the latter long sequence of ∞-categories to a long sequence of posets

$$\cdots \to \pi_2(\mathcal{F}, \bar{Z}) \to \pi_2(\mathcal{C}, Z) \to \pi_2(\mathcal{D}, \phi Z)$$

$$\to \pi_1(\mathcal{F}, \bar{Z}) \to \pi_1(\mathcal{C}, Z) \to \pi_1(\mathcal{D}, \phi Z)$$

$$\to \pi_0(\mathcal{F}, \bar{Z}) \to \pi_0(\mathcal{C}, Z) \to \pi_0(\mathcal{D}, \phi Z).$$

Let X be an ∞-category.
Recall that $\tau_{\leq 0}(X)$ is the set of equivalence classes of objects modulo the relation $A \simeq B$ if and only if there is a 0-connection between A and B: morphisms $A \to B$ and $B \to A$.

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Recall that $\tau_{\leq 1}(X)$ is a category enriched in posets. Two objects A, B in X are equivalent in $\tau_{\leq 1}(X)$ if and only if there is a 1-connection between A and B: morphisms $f : A \to B$ and $g : B \to A$ and 2-morphisms

$$\mathrm{id} \to gf, gf \to \mathrm{id}, \mathrm{id} \to fg, fg \to \mathrm{id}.$$

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$$\mathrm{id} \to gf, gf \to \mathrm{id}, \mathrm{id} \to fg, fg \to \mathrm{id}.$$

The functor $\tau_{\leq 1}(X) \to \tau_{\leq 0}(X)$ sends a morphism $f : A \to B$ to the relation $A \leq B$.

The following is [1, Theorem 4.2.7.]:

Theorem

Let $\phi : \mathcal{C} \to \mathcal{D}$ *be a functor,* $Z := (X_i, Y_i)_{i \geq 0}$ *an oriented base point of* $\mathcal{C}$*. Let* $\mathcal{F}$ *be the oriented right fiber of* ϕ *over* Y_0*. Let* $n \geq 0$*. We consider the long sequence of posets*

$$\cdots \to \pi_1(\mathcal{D}, \phi Z) \xrightarrow{\delta_0} \pi_0(\mathcal{F}, \bar{Z}) \to \pi_0(\mathcal{C}, Z) \to \pi_0(\mathcal{D}, \phi Z)$$

1. *An object of* $\pi_n(\mathcal{C}, Z)$ *belongs to the image of the map* $\pi_n(\mathcal{F}, \bar{Z}) \to \pi_n(\mathcal{C}, Z)$ *if and only if it belongs to the oriented fiber of the map* $\pi_n(\mathcal{C}, Z) \to \pi_n(\mathcal{D}, \phi Z)$*.*
2. *An object of* $\pi_n(\mathcal{F}, \bar{Z})$ *belongs to the image of the map* $\pi_{n+1}(\mathcal{D}, \phi Z) \to \pi_n(\mathcal{F}, \bar{Z})$ *if and only if its image in* $\pi_n(\mathcal{C}, Z) = \tau_{\leq 0}(\mathrm{Mor}^n_{\mathcal{C}}(Z))$ *is* X_{n+1} *in* $\tau_{\leq 1}(\mathrm{Mor}^n_{\mathcal{C}}(Z))$*.*
3. *An object of* $\pi_{n+1}(\mathcal{D}, \phi Z)$ *belongs to the image of the map* $\pi_{n+1}(\mathcal{C}, Z) \to \pi_{n+1}(\mathcal{D}, \phi Z)$ *if and only if its image in* $\pi_n(\mathcal{F}, \bar{Z}) = \tau_{\leq 0}(\mathrm{Mor}^n_{\mathcal{F}}(\bar{Z}))$ *is* X_{n+1} *in* $\tau_{\leq 1}(\mathrm{Mor}^n_{\mathcal{F}}(\bar{Z}))$*.*

Condition (1) refers precisely to oriented right fiber if n is even, and to oriented left fiber if n is odd.

Definition

Let $n \geq 0$.

1. An ∞-category X is 0-directed if for every pair of objects $A, B \in X$ we have that $A \simeq B$ if there are morphisms $A \to B$ and $B \to A$.
2. An ∞-category is $n+1$-directed if it is 0-directed and all morphism ∞-categories are n-directed.

An ∞-category X is 0-directed if and only if $\tau_{\leq 0}(X)$ is the set of equivalence classes of objects of X endowed with the partial order: $A \leq B$ if and only if there is a morphism $A \to B$ in X.

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Definition

An ∞-category is directed if it is n-directed for every $n \geq 0$.

The next is [1, Theorem 4.2.12.]:

Theorem

Let $0 \leq n \leq \infty$. A functor $X \to Y$ of directed ∞-categories is an n-equivalence if and only if for every $0 \leq m \leq n$ and $m-1$-dimensional oriented base point Z of X the induced map of partially ordered sets

$$\pi_m(X, Z) \to \pi_m(Y, \phi Z)$$

is an isomorphism.

Example

$n = 0$: A functor $X \to Y$ of directed ∞-categories is a 0-equivalence if and only if the induced map of posets $\pi_0(X) \to \pi_0(Y)$ is an isomorphism. This is because $\pi_0(X)$ is the set of equivalence classes since X is directed, and similar for Y.

Let $A, B \in X$ and $0 \leq m \leq n-1$ and Z a $m-1$-dimensional oriented base point of $\mathrm{Mor}_X(A, B)$. Let

$$\phi' : \mathrm{Mor}_X(A, B)) \to \mathrm{Mor}_Y(\phi(A), \phi(B))$$

be the induced functor. The $m-1$-dimensional oriented base point Z of $\mathrm{Mor}_X(A, B)$ corresponds to an m-dimensional oriented base point Z' of X.
The induced map of posets

$$\pi_m(\mathrm{Mor}_X(A, B), Z) \to \pi_m(\mathrm{Mor}_Y(\phi(A), \phi(B)), \phi' Z)$$

identifies with the induced map of posets

$$\pi_{m+1}(X, Z') \to \pi_{m+1}(Y, \phi Z').$$

Thus ϕ' is an equivalence if we assume the statement for $n-1$ (= dimension of oriented base point Z').
Induction over n proves the result.

Next we compute the fundamental posets of the oriented simplices and the oriented cubes [1, 4.3.]:

The n-th oriental (oriented simplex) is the gaunt and loopfree n-category

$$\Delta^n := \Delta^0 * \ldots * \Delta^0.$$

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The n-th oriented cube is the gaunt and loopfree n-category

$$\Box^n := \Box^1 \boxtimes \ldots \boxtimes \Box^1.$$

The n-th oriental (oriented simplex) is the gaunt and loopfree n-category

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The n-th oriented cube is the gaunt and loopfree n-category

$$\square^n := \square^1 \boxtimes \ldots \boxtimes \square^1.$$

The set of objects of Δ^n is the set $\{0 < \ldots < n\}$.

The partial order on $\pi_0(\Delta^n)$ is the poset $\{0 < \ldots < n\}$.

The n-th oriental (oriented simplex) is the gaunt and loopfree n-category

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$$\square^n := \square^1 \boxtimes \ldots \boxtimes \square^1.$$

The set of objects of Δ^n is the set $\{0 < \ldots < n\}$.

The partial order on $\pi_0(\Delta^n)$ is the poset $\{0 < \ldots < n\}$.

The set of objects of $\square^n$ is the set $\{0 < \ldots < 1\}^{\times n}$.

The partial order on $\pi_0(\square^n)$ is the poset $\{0 < \ldots < 1\}^{\times n}$, the product of the poset $\{0 < 1\}$.

Via the isomorphism of posets

$$\{0 < 1\}^{\times n} \simeq \{M \subset \{1, \ldots, n\}\}.$$

we write objects of $\square^n$ as subsets $M \subset \{1, \ldots, n\}$.

Lemma

Let X be a Steiner ∞-category and $A, B \in X$.
The poset $\pi_1(X, (A, B))$ is canonically isomorphic to the

$$\{A \to T_1 \to \ldots \to T_n \to B \mid n \geq 0\}$$

of sequences of atomic morphisms in X eqipped with the following partial order:

A sequence

$$A \to S_1 \to \ldots \to S_m \to B$$

is smaller or equal as a sequence

$$A \to T_1 \to \ldots \to T_n \to B$$

if and only if there is an atomic 2-morphism from a subsequence $S_{i_1} \to \ldots \to S_{i_k}$ to a subsequence $T_{j_1} \to \ldots \to T_{j_\ell}$.

Corollary

For each nondegenerate 1-cell $(i < j) : \mathbb{D}^1 \to \pi_0(\Delta^n)$, *there is a pullback square of the form*

$$\begin{array}{ccc} S((\mathbb{D}^1)^{\times(j-i-1)}) & \longrightarrow & \tau_{\leq 1}(\Delta^n) \\ \downarrow & & \downarrow \\ \mathbb{D}^1 & \xrightarrow{(i<j)} & \tau_{\leq 0}(\Delta^n). \end{array}$$

In particular, there are canonical equivalences

$$\pi_1(\Delta^n, (i,j)) = \mathbb{D}^{1\times(j-i-1)}$$

for all 0*-dimensional oriented basepoints* $(i,j) : \partial\mathbb{D}^1 \to \square^n$ *such that* $i < j$.

Proof.

The poset $\pi_1(\Delta^n, (i,j))$ is the poset of sequences

$$i < k_1 < \ldots < k_m < j$$

for $m \geq 1$ in $\pi_0(\Delta^n) = \{0 < \ldots < n\}$ corresponding to sequences of atomic morphisms.

There is a (necessarily) unique atomic morphism $a \to b$ in Δ^n if and only if $b = a + 1$.

$\implies \pi_1(\Delta^n, (i,j))$ is the poset of sequences

$$i < k_1 < \ldots < k_m < j$$

for $m \geq 1$ such that $k_{j+1} = k_j + 1$ for every $1 \leq k < m$.

This poset is precisely the poset

$$\{S \subset \{i+1, \ldots, j-1\}\} = (\mathbb{D}^1)^{\times(j-i-1)}.$$

Let $n \geq 0$.

Let $\mathbb{S}^n$ denote the set of permutations of the finite set $\{1, ..., n\}$ with n-elements.

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We equip $\mathbb{S}^n$ with a partial order generated by setting

$$\rho \leq \sigma$$

whenever

$$\sigma = \tau_{\rho(i),\rho(i+1)} \circ \rho$$

for some $1 \leq i \leq n$ such that $\rho(i) < \rho(i+1)$.

Corollary

Let $n \geq 0$. For each nondegenerate 1-cell $(M \subset N) : \mathbb{D}^1 \to \pi_0(\square^n)$, there is a pullback square of the form

$$\begin{array}{ccc} S(\mathbb{S}^{|N \smallsetminus M|}) & \longrightarrow & \tau_{\leq 1}(\square^n) \\ \downarrow & & \downarrow \\ \mathbb{D}^1 & \xrightarrow{M \subset N} & \tau_{\leq 0}(\square^n). \end{array}$$

In particular, there are canonical equivalences

$$\pi_1(\square^n, (M, N)) = \mathbb{S}^{|M \smallsetminus N|}$$

for all 0-dimensional oriented basepoints $(M, N) : \partial\mathbb{D}^1 \to \square^n$ such that $M \subset N$.

Proof.

The poset $\pi_1(\square^n, (M,N))$ is the poset of sequences $M \subset M_1 \subset \dots \subset M_m < N$ for $m \geq 1$ in $\pi_0(\square^n) = \{M \subset \{0, \dots, n\}\}$ corresponding to sequences of atomic morphisms.

In $\square^n$ there is a (necessarily) unique atomic morphism $M' \to M''$ if and only if $M \subset M''$ and moreover

$$|M'' \setminus M'| = 1.$$

$\implies$ The poset $\pi_1(\square^n, (M,N))$ is the poset of sequences $M \subsetneq M_1 \subsetneq \dots \subsetneq M_m \subsetneq N$ for $m \geq 1$ in $\pi_0(\square^n) = \{M \subset \{0, \dots, n\}\}$ such that $|M_{i+1} \setminus M_i| = 1$ for $1 \leq i < m$. This poset is precisely the poset $\mathbb{S}^{|N \setminus M|}$.

□

David Gepner and Hadrian Heine.
Homotopy posets, Postnikov towers, and hypercompletions of ∞-categories.
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