David Gepner.

16 March 2026.

FRG workshop.

Premise: lift homotopy theory to higher categorical world.

Think $\infty$-category $\big(= (\infty,\infty)\text{-category}\big)$,

$n$-category $\big(= (\infty,n)\text{-cat}\big)$,

as generalized spaces.

Any space $X$ can be obtained by gluing cells.

<u>Want</u>. • $sk_n X$ skeletal filtrations filtration on $\infty$-cat. (Colimit behavior)

•
$$\begin{tikzcd} & \mathrm{Mor}_X(s,t) \arrow[dl] \arrow[dr] & & (n-1)\mathrm{Cat} & \infty\mathrm{Cat} \\ * \arrow[dr] \arrow[rr, phantom, "\Rightarrow"] & & * \arrow[dl] & & \\ & X & & n\mathrm{Cat} & \infty\mathrm{Cat} \end{tikzcd}$$

oriented pullbacks give mapping spaces. (Limit behavior.)

$$\begin{tikzcd} \cdots\cdots \; 2\text{-Cat} \arrow[r, bend left, "\tau_1"] \arrow[r, leftarrow] \arrow[r, bend right, "\iota_1"'] & 1\text{-Cat} \arrow[r, bend left, "\tau_0 = \text{geo. real.}"] \arrow[r, leftarrow] \arrow[r, bend right, "\iota_0 = \text{groupoid core}"'] & 0\text{Cat} = \mathcal{S} \end{tikzcd}$$

$1\text{-Cat} = \mathcal{S}\text{-Cat}$

homotopy types
anima
$\infty$-groupoids

<u>Def</u>. $\infty\text{-Cat} = \lim_{\text{core functors}} n\text{Cat} = \lim\big(\cdots 2\text{-Cat} \xrightarrow{\iota_1} 1\text{-Cat} \xrightarrow{\iota_0} 0\text{-Cat}\big)$

$\in \mathrm{Pr}^R$. or in CAT.

<u>Rem</u>. Could also take the limit along the truncation functors $\tau_n$.
Like Postnikov towers, analogous to hypercomplete objects.

**Suspension functor.** $\quad S : n\mathrm{Cat} \longrightarrow (n+1)\mathrm{Cat}_{\partial\Delta^1/}$

$$S(X) = \{\, 0 \xrightarrow{\;X\;} 1 \,\}$$

In the limit, get $S : \infty\mathrm{Cat} \longrightarrow \infty\mathrm{Cat}_{\partial\Delta^1/}$.

Right adjoint $\quad \mathrm{Mor}_{-}(-,-) : \infty\mathrm{Cat}_{\partial\Delta^1/} \longrightarrow \infty\mathrm{Cat}$

$$(X,(s,t)) \longmapsto \mathrm{Mor}_X(s,t).$$

Use suspension to create higher-dim cats.

0-disk. $\quad \mathbb{D}^0 = *,\quad S(\mathbb{D}^0) = \mathbb{D}^1 = \{0 \to 1\}$

$$S(\mathbb{D}^1) = \mathbb{D}^2 = \left\{ \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd} \right\}$$

$$S(\mathbb{D}^2) = \mathbb{D}^3 = \left\{ \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \Rrightarrow \Downarrow"] & 1 \end{tikzcd} \right\}$$

$\vdots$

These are the points of $\infty\mathrm{Cat}$.

Jointly conservative.

$\partial\mathbb{D}^0 = \emptyset$

$S(\emptyset) = \{0,1\} = \partial\mathbb{D}^1$

$S(\partial\mathbb{D}^1) = \partial\mathbb{D}^2 = \left\{ \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] & 1 \end{tikzcd} \right\}$ (Kronecker quiver)

$\vdots$

$I_0\, \partial\mathbb{D}^n \simeq S^{n-1} \subseteq I_0\, \mathbb{D}^n \simeq D^n$.

$$\begin{tikzcd}
\partial\mathbb{D}^n \arrow[r] \arrow[d] & \mathbb{D}^n \arrow[d] \\
* \arrow[r] & \mathbb{S}^n
\end{tikzcd}$$

pushout square

categorical $n$-sphere.

generating degree $n$ endomorphism.

$$\Vert$$

$B^n \underline{\mathrm{Conf}(\mathbb{R}^n)}$ (Not the group completion.)

Need to categorify Cartesian product.

$X \times Y \simeq \coprod_X Y$ no longer suffices. Replace w/ Gray tensor product.

Combinatorics of Gray tensor product live on $Ch(\mathbb{Z})$.

Steiner theory.

$$\begin{array}{ccc}
 & & C_\cdot(\mathbb{D}^1) \\
 & & \Vert \\
\mathbb{D}^1 & \subseteq & \mathbb{Z}\cdot\{s,t\} \xleftarrow{\ d\ } \mathbb{Z}\cdot x \\
\Vert & & dx = t - s \\
\{s \xrightarrow{x} t\} & &
\end{array}$$

chain complex

$\mathbb{D}^1 \boxtimes \mathbb{D}^1 \subseteq C_\cdot(\mathbb{D}^1) \otimes_{\mathbb{Z}} C_\cdot(\mathbb{D}^1)$. Using usual sign convention.

$$\begin{tikzcd}
s\otimes s \arrow[r, "x\otimes s"] \arrow[d, "s\otimes x"'] \arrow[dr, phantom, "x\otimes x"] & t\otimes s \arrow[d, "t\otimes x"] \\
s\otimes t \arrow[r, "x\otimes t"'] & t\otimes t
\end{tikzcd}$$

$\Nearrow$ positive

$$\begin{aligned}
\partial(x\otimes x) &= \partial(x)\otimes x - x\otimes\partial(x) \\
&= (t-s)\otimes x - x\otimes(t-s) \\
&= t\otimes x + x\otimes s \\
&\quad - s\otimes x - x\otimes t,
\end{aligned}$$

"atomic morphisms"

Basis of atomic elements.

$\square^n = (\mathbb{D}^1)^{\boxtimes n}$, $\square \subseteq \infty\mathrm{Cat}$, $\left(\mathcal{P}(\square), \boxtimes\right) \rightleftarrows \left(\infty\mathrm{Cat}, \boxtimes\right)$.

Campion.

(This does not work for the $\mathbb{D}^n$.)
(Could glue to get $\Theta$s.)

$\Delta^n \subseteq C_\bullet(\Delta^n)$

$$\underset{\text{som}}{\left(\mathcal{P}(\Delta), \star\right)} \rightleftarrows \left(\infty\mathrm{Cat}, \star\right)$$

$$\Delta^2 = \left\{ \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3, "\Uparrow"] & & 2 \end{tikzcd} \right\}$$

$$\Delta^n \star \Delta^m = \Delta^{n+m+1}.$$

---

$X, Y, Z \in \infty\mathrm{Cat}$

$$\mathrm{Map}\left(X, \mathrm{Fun}^{\mathrm{oplax}}(Y,Z)\right) := \mathrm{Map}\left(X \boxtimes Y, Z\right) =: \mathrm{Map}\left(Y, \mathrm{Fun}^{\mathrm{lax}}(X,Z)\right)$$

$$\mathrm{Map}_{Y/}\left(X \star Y, Z\right) =: \mathrm{Map}\left(X, Z_{//g}\right), \quad \mathrm{Map}\left(Y, Z_{f//}\right) := \mathrm{Map}_{X/}\left(X \star Y, Z\right)$$

$$\emptyset \star Y = Y \xrightarrow{\text{can}} X \star Y, \quad Y \xrightarrow{g} Z \qquad\qquad X \star \emptyset = X \xrightarrow{\text{can}} X \star Y, \quad X \xrightarrow{f} Z$$

**Def.** An oriented category is an $(\infty\text{-Cat}, \overline{\boxtimes})\text{-Cat} = \mathrm{Cat}\text{-}(\infty\mathrm{Cat}, \boxtimes)$

$$X \overline{\boxtimes} Y = \left(X^{co} \boxtimes Y^{co}\right)^{co}$$

Left versus right enrichment.

"conjugate"

reverse even-dim cells.

**Example.** $\mathrm{Cyl}(X) = X \boxtimes \mathbb{D}^1$ endofunctor of $\infty\mathrm{Cat}$ viewed as a $1\mathrm{Cat}$.

Not compatible with $(\infty\mathrm{Cat}, \times)$-enrichment.

Check taking $X = \mathbb{D}^0$. Would get

$$
\begin{tikzcd}
\mathbb{D}^1 \times \mathrm{Cyl}(\mathbb{D}^0) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{Cyl}(\mathbb{D}^1 \times \mathbb{D}^0) \arrow[d, phantom, "\cong"] \\
\mathbb{D}^1 \times \mathbb{D}^1 \arrow[r, dashed, "\text{No!}"'] & \mathbb{D}^1 \boxtimes \mathbb{D}^1
\end{tikzcd}
$$

Does not commute.

**Join formula.**

$$
\begin{tikzcd}
X \boxtimes \partial\mathbb{D}^1 \boxtimes Y \arrow[r] \arrow[d] & X \boxtimes \mathbb{D}^1 \boxtimes Y \arrow[d] \\
X + Y \arrow[r] & X \star Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \boxtimes \partial\mathbb{D}^1 \arrow[r] \arrow[d] & X \boxtimes \mathbb{D}^1 \arrow[d] \\
\partial\mathbb{D}^1 \arrow[r] & S(X)
\end{tikzcd}
$$

pushout. pushout.

**Def.** $f : Y \to X$ is $n$-truncated if

$$
\begin{tikzcd}
\partial\mathbb{D}^m \arrow[r] \arrow[d] & Y \arrow[d] \\
\mathbb{D}^m \arrow[r] \arrow[ur, dashed, "\exists!"] & X
\end{tikzcd}
$$

$\exists!$ for all $m > n$.

Filling space is contractible.

$n$-connected left orthogonal of some factorization system.

An object $X$ of an oriented cat is $n$-truncated if $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is $n$-truncated $\forall Y$.
$\mathrm{Mor}_{\mathcal{C}}(X, Y) \in \infty\mathrm{Cat}$

Get Postnikov towers and skeletal filtrations.

Consider $\Delta^+ := \{\Delta^n, I_{n-1}\Delta^n\}_{n\in\mathbb{N}} \subset \infty\mathrm{Cat}$

and "atomic" maps between them.

"Complicial philosophy". $\mathcal{P}(\Delta^+) \rightleftarrows \infty\mathrm{Cat}$

Street – Roberts

Loubaton

Filter $\Delta^+$ by cardinality/dimension.

Get a filtration $X = \operatorname{colim} sk_n X$

best approximation by $\Delta^+_{\le n}$.

$$\begin{tikzcd} \coprod\limits_{I_n\Delta^n \to X} \partial\mathbb{D}^n + \coprod\limits_{\Delta^{n+1}\to X} \partial\mathbb{D}^{n-1} \arrow[r] \arrow[d] & sk_{n-1}X \arrow[d, hook] \\ \coprod \mathbb{D}^{n-1} + \coprod \mathbb{D}^n \arrow[r] & sk_n X \end{tikzcd}$$

$\coprod \mathbb{D}^{n-1} = I_{n-1}\mathbb{D}^n$

Can be used to give an obstr. theory for

$$\begin{tikzcd} sk_{n-1}X \arrow[r] \arrow[d] & Y \\ sk_n X \arrow[ur, dashed] \end{tikzcd}$$

$$sk_n X / sk_{n-1} X = (\vee S^n) \vee (\vee \mathbb{S}^n)$$

<u>Def.</u> A morphism $p: Y \to X$ in $\infty$-Cat is a coCart fibration if it is classified

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \{\mathbb{D}^0\} \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ X \arrow[r, "\exists f"'] & \infty\mathrm{Cat}. \end{tikzcd}$$

Or

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \infty\mathrm{Cat}_{\mathbb{D}//} \arrow[d] \\ X \arrow[r] & \infty\mathrm{Cat} \end{tikzcd}.$$