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FRG workshop.

Premise: lift homotopy theory to higher categorical world.

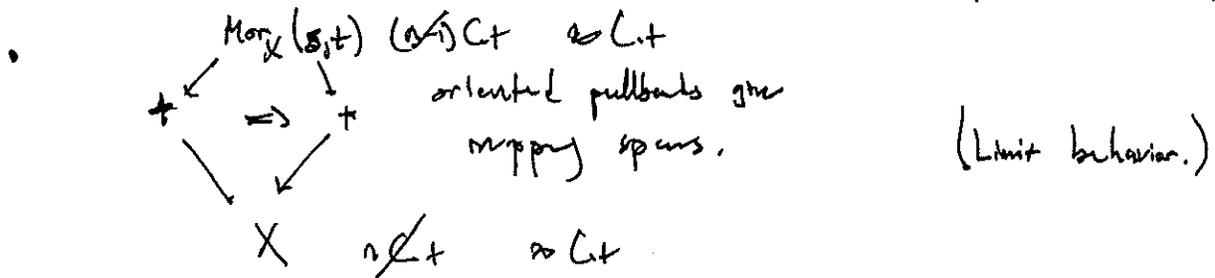
Think ∞ -category (= (\mathcal{S}, ∞) -category),

n -category (= (\mathcal{S}, n) -cat)

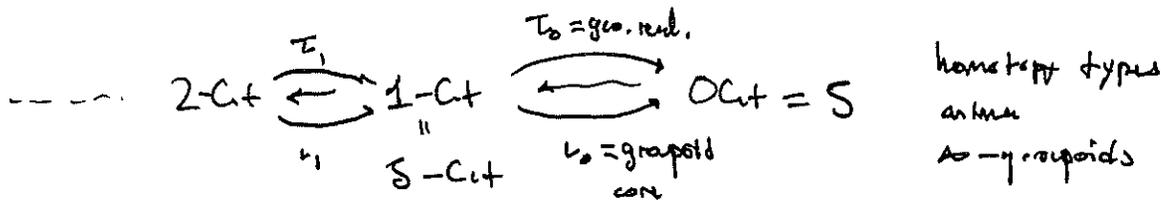
as generalized spaces.

Any space X can be obtained by gluing cells.

Want. • skel X skeletal filtration filtration on ∞ -cat. (Limit behavior.)



(Limit behavior.)



Def. ∞ -cat = $\lim_{\text{core functors}} n\text{-Cat} = \lim (\dots 2\text{-Cat} \xrightarrow{v_1} 1\text{-Cat} \xrightarrow{v_0} 0\text{-Cat})$

$\in \mathcal{P}_r^R$. or in CAT.

Rem. Could also take the limit along the truncation functors T_n .

Like Postnikov tower, analogous to hypercomplete objects.

Suspension Functor. $S: n\text{Cat} \longrightarrow (n+1)\text{Cat}$
 $\partial\Delta^1$

$$S(X) = \left\{ 0 \xrightarrow{X} 1 \right\}$$

In the limit, get $S: \infty\text{Cat} \longrightarrow \infty\text{Cat}$
 $\partial\Delta^1$

Right adjoint $\text{Mor}_-(-, -): \infty\text{Cat}_{\partial\Delta^1} \longrightarrow \infty\text{Cat}$

$$(X, (s, t)) \longmapsto \text{Mor}_X(s, t).$$

Use suspension to create higher-dim cats.

0-disk. $\mathbb{D}^0 = *$, $S(\mathbb{D}^0) = \mathbb{D}^1 = \{0 \rightarrow 1\}$

$$S(\mathbb{D}^1) = \mathbb{D}^2 = \left\{ \begin{array}{c} 0 \quad 1 \\ \downarrow \\ 0 \quad 1 \end{array} \right\}$$

$$S(\mathbb{D}^2) = \mathbb{D}^3 = \left\{ \begin{array}{c} 0 \quad 1 \\ \downarrow \\ 0 \quad 1 \\ \downarrow \\ 0 \quad 1 \end{array} \right\}$$

These are the points of ∞Cat .

Jointly conservative.

$$\partial\mathbb{D}^0 = \emptyset$$

$$S(\emptyset) = \{0, 1\} = \partial\mathbb{D}^1$$

$$S(\partial\mathbb{D}^1) = \partial\mathbb{D}^2 = \left\{ 0 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} 1 \right\} \quad (\text{Kronecker quiver})$$

$$\tau_0 \partial\mathbb{D}^n \cong S^{n-1} \subseteq \tau_0 \mathbb{D}^n = \mathbb{D}^n.$$

$$\begin{array}{ccc} \partial \mathbb{D}^n & \longrightarrow & \mathbb{D}^n \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^n \end{array} \quad \begin{array}{l} \text{product space} \\ \\ \text{categorical } n\text{-sphere.} \end{array}$$

generating // degree n endomorphism.

$$B^n \text{ Conf}(\mathbb{R}^n) \quad (\text{Not the group completion.})$$

Need to categorify Cartesian product.

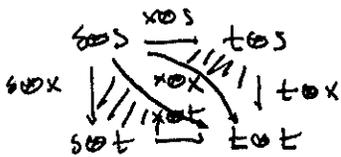
$$X \times Y \cong \coprod_x Y \quad \text{no longer suffices. Replace w/ Gray tensor product.}$$

Contributors of gray tensor product live on $\text{Ch}(\mathbb{Z})$.

Stellar theory.

$$\begin{array}{c} \mathbb{D}^1 \\ \parallel \\ \{s \xrightarrow{x} t\} \end{array} \in \begin{array}{c} C.(\mathbb{D}^1) \\ \cong \\ \mathbb{Z} \cdot \{s, t\} \xleftarrow{d} \mathbb{Z} \cdot x \\ \downarrow dx = t - s \end{array} \quad \text{chain complex}$$

$$\mathbb{D}^1 \boxtimes \mathbb{D}^1 \in C.(\mathbb{D}^1) \otimes_{\mathbb{Z}} C.(\mathbb{D}^1). \quad \text{Very usual sign convention.}$$



$$\begin{aligned} \partial(x \otimes x) &= \partial(x) \otimes x - x \otimes \partial(x) \\ &= (t-s) \otimes x - x \otimes (t-s) \\ &= t \otimes x + x \otimes s \\ &\quad - s \otimes x - x \otimes t, \end{aligned}$$

"atomic morphisms"

Basis of atomic elements.

$$\mathbb{D}^\wedge = (\mathbb{D}')^{\boxtimes n}, \quad \mathbb{D} \in \infty\text{Cat}, \quad (\mathcal{P}(\mathbb{D}), \boxtimes) \rightleftarrows (\infty\text{Cat}, \boxtimes).$$

Compton.

(This does not work ~~for~~ for the \mathbb{D}' .)
 (Could give to get \mathbb{D} 's.)

$$\Delta^n \in C.(\Delta^n) \quad (\mathcal{P}(\Delta), \star_{\text{JOM}}) \rightleftarrows (\infty\text{Cat}, \star)$$

$$\Delta^2 = \left\{ \begin{array}{ccc} & 1 & \\ \swarrow & \uparrow & \searrow \\ 0 & & 2 \end{array} \right\}$$

$$\Delta^n \star \Delta^m = \Delta^{n+m+1}$$



$$X, Y, Z \in \infty\text{Cat}$$

$$\text{Map}(X, \text{Fun}^{\text{opLix}}(Y, Z)) := \text{M.p.}(X \boxtimes Y, Z) =: \text{M.p.}(Y, \text{Fun}^{\text{Lix}}(X, Z))$$

$$\text{M.p.}_{Y|} (X \star Y, Z) =: \text{M.p.}(X, Z_{||g}), \quad \text{M.p.}(Y, Z_{||f}) =: \text{M.p.}_{X|} (X \star Y, Z)$$

$\begin{array}{ccc} \text{can} / & & / g \\ \phi \star Y = Y & & \end{array}$

$\begin{array}{ccc} \text{can} \uparrow & & \uparrow f \\ X \star \phi = X & & \end{array}$

Def. An oriented category is an $(\infty\text{-Cat}, \bar{\boxtimes})\text{-Cat} = \text{Cat} - (\infty\text{Cat}, \bar{\boxtimes})$

Left words
right enrichment.

$$X \bar{\boxtimes} Y = (X^{\text{co}} \boxtimes Y^{\text{co}})^{\text{co}}$$

"conjugate"

reverse even-dim cells.

Example. $\text{Cyl}(X) = X \boxtimes \mathbb{D}^1$ endofunctor of ∞Cat viewed as a 1Cat .

Not compatible with $(\infty\text{Cat}, \times)$ -enrichment.

Check that $X = \mathbb{D}^0$. Would get

$$\begin{array}{ccc} \mathbb{D}^1 \times \text{Cyl}(\mathbb{D}^0) & \xrightarrow{\quad} & \text{Cyl}(\mathbb{D}^1 \times \mathbb{D}^0) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{D}^1 \times \mathbb{D}^1 & \xrightarrow{\quad} & \mathbb{D}^1 \boxtimes \mathbb{D}^1 \\ & \text{No!} & \end{array}$$

Does not commute.

John formula

$$\begin{array}{ccc} X \boxtimes \mathbb{D}^1 \boxtimes Y & \xrightarrow{\quad} & X \boxtimes \mathbb{D}^1 \boxtimes Y \\ \downarrow & & \downarrow \\ X + Y & \xrightarrow{\quad} & X + Y \\ \text{pushout.} & & \end{array}$$

$$\begin{array}{ccc} X \boxtimes \mathbb{D}^1 & \xrightarrow{\quad} & X \boxtimes \mathbb{D}^1 \\ \downarrow & & \downarrow \\ \mathbb{D}^1 & \xrightarrow{\quad} & S(X) \\ \text{pushout.} & & \end{array}$$

D.F. $f: Y \rightarrow X$ is n -truncated if

n -connected left orthogonal of some factorization system.

$$\begin{array}{ccc} \mathbb{D}^n & \xrightarrow{\quad} & Y \\ \downarrow & \dashrightarrow^{\exists!} & \downarrow \\ \mathbb{D}^n & \xrightarrow{\quad} & X \end{array}$$

An object X of an oriented cat is n -truncated if $\text{Mor}_{\infty\text{Cat}}(X, Y)$ is n -truncated $\forall Y$.

$\exists!$ For all $m \geq n$.
~~Full~~ ~~is~~ ~~contractible~~.
 Full space is contractible.

Get Postnikov towers and skeletal fibrations.

Consider $\Delta^+ := \{ \Delta^n, \tau_{n-1} \Delta^n \}_{n \in \mathbb{N}} \subset \infty \text{Cat}$

and "atomic" maps between them.

"Complicated philosophy".

$$\mathcal{P}(\Delta^+) \begin{matrix} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{matrix} \infty \text{Cat}$$

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Loubaton

Filter Δ^+ by cardinality / dimension.

Get a fibration $X = \text{colim } \text{sk}_n X$

best approximation by Δ_{cell}^+ .

$$\begin{array}{ccc} \coprod_{\tau_{n-1} \Delta^n} \mathbb{Z} \mathbb{D}^n & + & \coprod_{\Delta^{n+1} \rightarrow X} \mathbb{Z} \mathbb{D}^{n+1} & \rightarrow & \text{sk}_{n+1} X \\ \downarrow & & & & \downarrow \\ \coprod_{\Sigma_{n-1} \mathbb{D}^n} \mathbb{Z} \mathbb{D}^{n-1} & + & \coprod \mathbb{Z} \mathbb{D}^n & \rightarrow & \text{sk}_n X \end{array}$$

Can be used to give an obstr. theory

$$\begin{array}{ccc} \text{sk}_{n+1} X & \longrightarrow & Y \\ \downarrow & \nearrow & \\ \text{sk}_n X & & \end{array}$$

$$\text{sk}_n X / \text{sk}_{n-1} X = (V S^n) \vee (V \mathbb{S}^n)$$

Def. A morphism $p: Y \rightarrow X$ in $\infty\text{-cat}$ is a coCart fibration

~~if~~ if it is classified

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & \mathbb{Z} \mathbb{D}^n \\ \downarrow & \nearrow & \downarrow \\ X & \xrightarrow{\quad} & \infty \text{Cat} \\ & \exists f & \end{array}$$

Or

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & \infty \text{Cat} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad} & \infty \text{Cat} \end{array}$$