Hodge Numbers of Derived-Equivalent Complex Fourfolds

A self-contained working note

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Abstract

We prove that derived-equivalent smooth projective complex fourfolds have the same Hodge numbers. The proof has two ingredients. First, a four-dimensional Hodge–Riemann argument shows directly that $H^*(X, \mathcal{O}_X)$ is determined by $\mathrm{D^b}(X)$. Second, the symmetrized Mukai pairing shows that the ordinary topological signature is a derived invariant. Hochschild–Kostant–Rosenberg then leaves only one possible change in the Hodge diamond, and the signature eliminates it.

The withdrawn preprint claiming invariance of all $h^{0,p}$ in arbitrary dimension is not used. All details needed in dimension four are included below.

1 Statement

Theorem 1. *Let X and Y be smooth projective varieties of dimension four over $\mathbf{C}$. If*

$$\mathrm{D^b}(X) \simeq \mathrm{D^b}(Y)$$

as $\mathbf{C}$-linear triangulated categories, then

$$h^{p,q}(X) = h^{p,q}(Y) \qquad (0 \le p, q \le 4).$$

The proof of the edge Hodge numbers is given in Sections 2 and 3. Section 4 proves the signature invariance needed to determine the remaining diagonal Hodge numbers. The final bookkeeping is in Section 5.

2 A fourfold positivity lemma

For a smooth projective fourfold Z, put

$$\mathrm{Hdg}^*(Z) = \bigoplus_{j=0}^{4} (H^{2j}(Z, \mathbf{Q}) \cap H^{j,j}(Z)).$$

Write $v = \sum_{j=0}^{4} v_j$, where v_j has cohomological degree $2j$, and set

$$v^\vee = \sum_{j=0}^{4} (-1)^j v_j.$$

The Euler form on $\mathrm{Hdg}^*(Z)$ is

$$\chi_Z(v, w) = \int_Z v^\vee w \ \mathrm{td}(Z).$$

Lemma 2 (Fourfold positivity)**.** *Let* $0 \neq v \in \mathrm{Hdg}^*(Z)$*. If*

$$\chi_Z(\mathrm{ch}\, L, v) = 0 \qquad \textit{for every } L \in \mathrm{Pic}(Z),$$

then $\chi_Z(v, v) > 0$*.*

Proof. Write

$$\tau = \mathrm{td}(Z), \qquad w = v\tau = \sum_{j=0}^{4} w_j, \qquad w_j \in H^{2j}(Z, \mathbf{Q}) \cap H^{j,j}(Z).$$

Multiplication by τ is invertible, so $w \neq 0$.

Step 1: extract the homogeneous relations. For $D = c_1(L)$, the hypothesis is

$$0 = \int_Z e^{-D} w. \tag{2.1}$$

By the Lefschetz $(1,1)$ theorem, choose integral divisor classes $D_1, \ldots, D_\rho$ which form a rational basis of the rational $(1,1)$-classes. Applying (2.1) to tensor products of integer powers of the corresponding line bundles shows that

$$F(m_1, \ldots, m_\rho) = \sum_{r=0}^{4} \frac{(-1)^r}{r!} \int_Z \left(\sum_i m_i D_i \right)^r w_{4-r}$$

vanishes for every $(m_1, \ldots, m_\rho) \in \mathbf{Z}^\rho$. A polynomial over $\mathbf{Q}$ which vanishes on all of $\mathbf{Z}^\rho$ is zero. Hence every homogeneous part of F is zero. Polarizing the degree-r part gives

$$\int_Z D_1 \cdots D_r \, w_{4-r} = 0 \quad (0 \leq r \leq 4) \tag{2.2}$$

for all rational divisor classes D_i.

Step 2: determine the components of w. Fix an ample rational divisor class A. We will use twice the following standard consequence of hard Lefschetz and the Hodge index theorem:

$$H^2(Z, \mathbf{Q}) \cap H^{1,1}(Z) \;\times\; H^6(Z, \mathbf{Q}) \cap H^{3,3}(Z) \longrightarrow \mathbf{Q}, \qquad (D, \Gamma) \longmapsto \int_Z D\Gamma$$

is a perfect pairing. Equivalently, the form $(D, E) \mapsto \int_Z A^2 DE$ is nondegenerate on rational $(1,1)$-classes.

We now read (2.2) one value of r at a time.

- For $r = 4$, take all four divisors equal to A. Since $\int_Z A^4 > 0$, the relation $\int_Z A^4 w_0 = 0$ gives $w_0 = 0$.

- For $r = 3$, take the divisors to be A, A, E. Then $\int_Z A^2 E w_1 = 0$ for every rational $(1,1)$-class E. Nondegeneracy of the preceding form gives $w_1 = 0$.

- For $r = 2$, take the divisors to be A, E. Then $\int_Z E(Aw_2) = 0$ for every rational $(1,1)$-class E. Here Aw_2 is a rational $(3,3)$-class, so perfectness of the displayed pairing gives

$$Aw_2 = 0.$$

  This is exactly the statement that $w_2 \in H^4(Z)$ is primitive.

- For $r=1$, we have $\int_Z Dw_3 = 0$ for every rational $(1,1)$-class D. The same perfect pairing gives $w_3 = 0$.
- For $r=0$, we have $\int_Z w_4 = 0$. Since $H^8(Z,\mathbf{Q}) \cap H^{4,4}(Z) = \mathbf{Q}[\mathrm{pt}]$, this gives $w_4 = 0$.

Thus $w = w_2$, and w_2 is a nonzero primitive rational $(2,2)$-class.

Step 3: compute the Euler square and its sign. Since $v = w_2\tau^{-1}$, its components of codimension less than two vanish and $v_2 = w_2$. To obtain codimension four in $v^\vee v\tau$, the only possible choice of codimensions is $2+2+0$. Moreover, $v_2^\vee = (-1)^2 v_2 = v_2$ and $\tau_0 = 1$. Consequently

$$\chi_Z(v,v) = \int_Z w_2^2.$$

For a primitive class of type $(2,2)$ in a fourfold, the sign in the Hodge–Riemann bilinear relations is

$$(-1)^{4\cdot 3/2} i^{2-2} = 1.$$

Thus the intersection form is positive definite on primitive real $(2,2)$-classes. Since $w_2 \neq 0$, we conclude that $\int_Z w_2^2 > 0$, as required. □

Proposition 3 (A nonzero-rank transform)**.** *Let $\Phi\colon \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(Y)$ be an equivalence of smooth projective complex fourfolds. Some line bundle L on X satisfies*

$$\operatorname{rk}\Phi(L) \neq 0.$$

Proof. Fix a closed point $y \in Y$ and put $E = \Phi^{-1}(\mathcal{O}_y)$. If every $\Phi(L)$ had rank zero, then

$$0 = \operatorname{rk}\Phi(L) = \chi_Y(\Phi(L), \mathcal{O}_y) = \chi_X(L, E) \tag{2.3}$$

for every line bundle L on X.

Let $v = \mathrm{ch}(E)$. If $v = 0$, Hirzebruch–Riemann–Roch would give

$$\chi_X(\Phi^{-1}(\mathcal{O}_Y), E) = 0,$$

whereas equivalence and $\chi_Y(\mathcal{O}_Y, \mathcal{O}_y) = 1$ make the same number equal to 1. Thus $v \neq 0$. Lemma 2 and (2.3) imply

$$\chi_X(E,E) > 0.$$

On the other hand,

$$\chi_X(E,E) = \chi_Y(\mathcal{O}_y, \mathcal{O}_y) = \sum_{i=0}^{4} (-1)^i \binom{4}{i} = 0,$$

a contradiction. □

3 The edge Hodge numbers

Proposition 4. *Under the hypotheses of Theorem 1, there is an isomorphism of graded algebras*

$$H^*(X, \mathcal{O}_X) \simeq H^*(Y, \mathcal{O}_Y).$$

In particular, $h^{0,q}(X) = h^{0,q}(Y)$ for every q.

Proof. Choose L as in Proposition 3 and put $F = \Phi(L)$. For a perfect complex of nonzero rank, the unit and trace maps

$$\mathcal{O}_Y \longrightarrow \mathbf{R}\mathcal{H}om_Y(F,F) \longrightarrow \mathcal{O}_Y$$

compose to multiplication by $\mathrm{rk}(F)$. Over $\mathbf{C}$ this scalar is invertible, so the unit is split. On cohomology it gives an injective homomorphism of graded algebras

$$H^*(Y,\mathcal{O}_Y) \hookrightarrow \mathrm{Ext}^*_Y(F,F) \simeq \mathrm{Ext}^*_X(L,L) = H^*(X,\mathcal{O}_X).$$

Applying the same argument to Φ^{-1} gives an injection in the opposite direction. The graded pieces are finite dimensional, so the first injection is degreewise bijective and hence is the desired graded-algebra isomorphism. □

Remark 5. The preceding low-dimensional strategy is also recorded in Addington–Bragg, Appendix A. It is reproduced here in full so that no assertion from the withdrawn all-dimensional preprint is required. The indispensable dimension-four input is the positivity in Lemma 2; the same argument does not extend formally to arbitrary dimension.

4 Derived invariance of the signature

We recall the generalized Mukai pairing in the convention of Huybrechts. If $\alpha = \sum_j \alpha_j \in H^*(Z,\mathbf{C})$ with $\alpha_j \in H^j(Z,\mathbf{C})$, set

$$\alpha^\vee = \sum_j i^j \alpha_j.$$

Then

$$\langle \alpha, \beta \rangle_Z = \int_Z e^{c_1(Z)/2} \alpha^\vee \beta. \tag{4.1}$$

By Orlov representability, every equivalence here is of Fourier–Mukai type. Its cohomological transform is an isometry for (4.1), by Grothendieck–Riemann–Roch or, equivalently, by functoriality and adjointness of the Mukai pairing. We use only real even cohomology.

Proposition 6. *If smooth projective complex varieties X and Y of even complex dimension are derived equivalent, then their ordinary topological signatures are equal.*

Proof. Write $n = \dim X = \dim Y$, where equality follows by comparing the Serre functors, and assume n even. On real even cohomology, symmetrize (4.1):

$$S_Z(\alpha,\beta) = \langle \alpha,\beta\rangle_Z + \langle \beta,\alpha\rangle_Z.$$

For even classes α, β, multiplicativity of $^\vee$ and $(\alpha^\vee)^\vee = \alpha$ give

$$\left(e^{-c_1(Z)/2}\alpha^\vee\beta\right)^\vee = e^{c_1(Z)/2}\beta^\vee\alpha.$$

On a class of top cohomological degree $2n$, the operation $^\vee$ acts by $i^{2n} = (-1)^n$. It follows that

$$\langle \beta,\alpha\rangle_Z = (-1)^n \int_Z e^{-c_1(Z)/2}\alpha^\vee\beta.$$

Because n is even, we obtain

$$S_Z(\alpha,\beta) = \int_Z 2\cosh\left(\frac{c_1(Z)}{2}\right)\alpha^\vee\beta. \tag{4.2}$$

Let
$$a_Z = \sqrt{2\cosh(c_1(Z)/2)}.$$
This is a finite cohomological power series with nonzero constant term. Its components have degrees divisible by four, so $a_Z^\vee = a_Z$. Thus (4.2) says that S_Z is congruent, by multiplication by a_Z, to
$$Q_Z(\alpha,\beta) = \int_Z \alpha^\vee \beta. \tag{4.3}$$
The Fourier–Mukai isometry preserves S, so the real signatures of Q_X and Q_Y agree.

Poincaré duality decomposes the contribution to (4.3) from $H^{2k}(Z,\mathbf{R})\oplus H^{2n-2k}(Z,\mathbf{R})$, $2k \neq n$, into hyperbolic blocks. These blocks have signature zero. On the middle cohomology $H^n(Z,\mathbf{R})$, the form Q_Z is $(-1)^{n/2}$ times the ordinary intersection form. Thus the signature of Q_Z is $(-1)^{n/2}\sigma(Z)$. Equality for X and Y gives $\sigma(X) = \sigma(Y)$. □

5 Finishing the Hodge-number calculation

Let
$$\delta^{p,q} = h^{p,q}(X) - h^{p,q}(Y).$$
Proposition 4, Hodge symmetry, and Serre duality give $\delta^{p,q} = 0$ on the boundary of the Hodge diamond.

Derived equivalence preserves Hochschild homology. The Hochschild–Kostant–Rosenberg isomorphism therefore gives, for every integer k,
$$\sum_{p-q=k} \delta^{p,q} = 0. \tag{5.1}$$
For $k = 2$, equation (5.1) shows that $\delta^{1,3} = \delta^{3,1} = 0$. For $k = 1$, Hodge symmetry and Serre duality make the two possibly nonzero summands equal, so $\delta^{1,2} = \delta^{2,1} = \delta^{2,3} = \delta^{3,2} = 0$. On the main diagonal, write
$$t = \delta^{1,1} = \delta^{3,3}.$$
The case $k = 0$ of (5.1) gives
$$\delta^{2,2} = -2t. \tag{5.2}$$
Thus (5.2) is the only formal ambiguity left.

For a smooth projective complex fourfold, Hirzebruch's Hodge-theoretic signature formula is
$$\sigma(Z) = \sum_{p,q} (-1)^q h^{p,q}(Z). \tag{5.3}$$
Substituting (5.2) into (5.3) gives
$$\sigma(X) - \sigma(Y) = -t - 2t - t = -4t.$$
Proposition 6 makes the left side zero, so $t = 0$. Every $\delta^{p,q}$ vanishes, proving Theorem 1.

6 Formal-verification boundary

The companion `FourfoldHodgeBookkeeping.lean` contains no `sorry` and declares no problem-specific axioms. Checked by Lean 4.28.0 with Mathlib 4.28.0, it reconstructs the full 5-by-5 difference matrix from the nine Hodge/Serre orbits, computes the three relevant HKR diagonal sums and the signature sum, proves that the matrix vanishes, and checks the contradiction in Proposition 3.

This is not an end-to-end formalization of Theorem 1: Mathlib lacks the needed Fourier–Mukai, HKR, hard Lefschetz, Hodge–Riemann, and Hirzebruch-signature theories. Those geometric inputs remain ordinary proofs rather than being hidden as Lean axioms.

7 What this does and does not prove

The argument proves the Hodge-number conjecture for smooth projective complex varieties through dimension four. It does not prove the general conjecture. Two genuinely dimension-specific features occur: the primitive $(2, 2)$ positivity used to find a nonzero-rank transform, and the fact that after the edge Hodge numbers are fixed, HKR plus the signature removes the sole remaining fourfold ambiguity.

References

[1] N. Addington and D. Bragg, *Hodge numbers are not derived invariants in positive characteristic*, arXiv:2106.09949v2, 2025; Appendix A.

[2] A. Căldăraru and S. Willerton, *The Mukai pairing, I: a categorical approach*, New York J. Math. **16** (2010), 61–98.

[3] D. Huybrechts, *Fourier–Mukai Transforms in Algebraic Geometry*, Oxford Mathematical Monographs, Oxford University Press, 2006; Chapter 5.

[4] D. Orlov, *Equivalences of derived categories and $K3$ surfaces*, J. Math. Sci. **84** (1997), 1361–1381.