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FRG Workshop.

Toward a universal property of the Gray tensor product.

Joint w/ Crossen, Keidar, Moser, Na...

Slogan: things get simpler @ infinity.

__Examples.__ • Higher affineness. X non-affine, but 1-affine for qcqs schemes.

An analogy: things get simpler at $(\infty,1)$ compared to $(n,1)$.

__Ex.__ For topoi. $\mathfrak{X}^{op} \longrightarrow Cat$, $X \longmapsto \mathfrak{X}_{/X}$ preserves limits

__Problem.__ Things don't actually seem simpler; they seem more complicated.

② ①

__E.g.__ $\infty Cat = \lim_n nCat$

$\simeq \operatorname{colim}_n nCat$ in Pr^L

Solution? Systematic way of understanding how mCats and nCats interact.

Try to use $\boxtimes$.

But, ①, $\boxtimes$ seems complicated!

__Examples of relations.__

① $nCat_* \xhookleftarrow{} (n+1)Cat_{**}$

② $\Sigma : nCat \longrightarrow (n+1)Cat$.

These are unary operations.

ΣC $\quad 0 \xrightarrow{C} 1$

③ $mCat \times nCat \xrightarrow{\boxtimes} (m+n)Cat$

④ $\xrightarrow{\star} (m+n+1)Cat$

interdefinable

⊠ seems complicated.

<u>Ex</u>. Complicial ⊠ (Verity) complicated

Variation where things are simpler. Conical model (Campion, Doherty, Kapulkin, M...).

Conical. ~~A~~ Easier marking rules.

$$\left(A \otimes B, \ \begin{array}{l} a\otimes b \text{ marked if} \\ a \text{ or } b \text{ is marked} \end{array} \right).$$

What about <u>strictifying</u> in $(\infty,1)$-cats?

a. Naturally ∞-cat.

b. Simple.

<u>Approach</u> (C.–N., C.). Campbell–Wheeler <u>Steiner</u>.

Import ⊠ from a strict world, à la Steiner.

$$\begin{array}{ccc} P(aDc) & \longrightarrow & \infty Cat \\ \uparrow & & \uparrow \\ (\text{"augmented directed complexes"}, \otimes) & \longrightarrow & \infty Cat_{gaunt} \end{array}$$

strong Steiner

Gray t.p.

"strict (∞,∞)-cats"

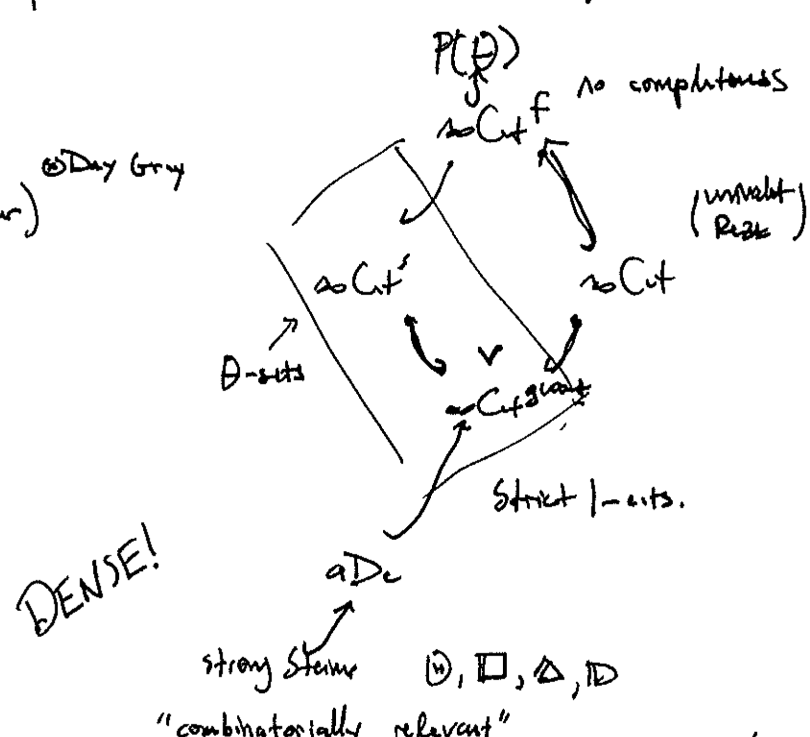
No non-triv isomorphisms.

$Hom(\mathbb{D}^n, C)$ is 0-truncated $\forall n$.

<u>Thm</u> 1)

$$\begin{array}{c} \text{Gray } P(\text{strong Steiner})^{\otimes \text{Day Gray}} \\ \downarrow \uparrow \\ \infty Cat. \end{array}$$

local $\simeq$s are closed under $\otimes$Day Gray.



Import even less?

Q. But what is $\boxtimes$, really?

Conjecture. Let $(A, \otimes, \mathbb{1}) \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Pr})$.

$$\begin{tikzcd}
\mathrm{Fun}^{L,\mathbb{E}_1}\left(\cancel{\infty\mathrm{Cat}^{\mathrm{Gray}}}\ \left(\prod_n n\mathrm{Cat}\right)^{\mathrm{Gray}}, A\right) \arrow[r] \arrow[d, shift left] & \mathrm{Fun}^L(2\mathrm{Cat}, A) \arrow[d, shift left] \\
\mathrm{Fun}^L(1\mathrm{Cat}\quad A) \arrow[r] \arrow[u, shift left] & \mathrm{Fun}^L(1\mathrm{Cat} \otimes \mathrm{Cat}, A). \arrow[u, shift left]
\end{tikzcd}$$

$2\mathrm{Cat}$ G-R $\to$ double cats

$2\mathrm{Cat} \leftarrow 1\mathrm{Cat}^{\otimes 2}$ This is some low-dim $\otimes$.

double

$$1\mathrm{Cat} \xrightarrow{\Gamma} A \ \longmapsto \ \left(\begin{tikzcd} 1\mathrm{Cat}^{\otimes 2} \arrow[r] & A^{\otimes 2} \arrow[d] \\ & A \end{tikzcd}\right)$$

Finite amt of data.

Can put $\infty\mathrm{Cat}^{\mathrm{gray}}$ in upper left,
but put $\mathbb{E}_0$ everywhere else.

Notes. (i) An oriented category $\mathcal{C}$ is $\simeq$ a module $\mathcal{C}$ over $\infty\mathrm{Cat}^{\mathrm{gray}}$ $\simeq$ ~~Fa~~ functor $\infty\mathrm{Cat}^{\mathrm{gray}} \xrightarrow{\Phi} \mathrm{End}(\mathcal{C})$

and $1\mathrm{Cat} \xrightarrow[L]{\Gamma, \mathbb{E}_0} \mathrm{End}(\mathcal{C})$

$\simeq$ $1\mathrm{Cat}^{\otimes 2} \xrightarrow[L]{2F(\Gamma)\ \mathbb{E}_0} \mathrm{End}(\mathcal{C})$

version of Gray tensor product.

Lawson-Reit $\Gamma : M \to M$ id cylinder...

$\Gamma^{\mathbb{L}^{op}_{\Phi}}$ Grattegory-Rozenblyum: $M \to M$ dual to composition.

(ii) $\mathrm{Cat}^{\otimes 2} \rightleftarrows 2\mathrm{Cat}$ conj.

2-cat of companions

$\mathcal{C}$ $2\mathrm{Cat}$

$\mathrm{Sq}\,\mathcal{C}$ 0-morphisms as in $\mathcal{C}$
hor, ver 1-morphisms as in $\mathcal{C}$
squares.

$$\begin{tikzcd} \bullet \arrow[r, "F'"] & \bullet \\ \bullet \arrow[u, "G"] \arrow[r, "F"'] \arrow[ur, phantom, "\Searrow"] & \bullet \arrow[u, "G'"'] \end{tikzcd}$$

Frame$_2$ squares.