Singularities in mixed characteristic via Riemann-Hilbert correspondence

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Introduction

Goal: Measure mildness of singularities.

Setting: $X = \mathrm{Spec}(R)$ - finite type normal affine scheme over a DVR

Characteristic zero: use projective birational maps; e.g.

X has *rational singularities* $\overset{def}{\iff}$ $R\pi_*\mathcal{O}_Y = \mathcal{O}_X$ for some resolution of singularities $\pi\colon Y \to X$.

Positive characteristic: use Frobenius; e.g.

- X is regular $\iff$ Frobenius is flat (Kunz)
- X has *mild* singularities $\iff$ $F^*\colon \mathcal{O}_X \to F_*\mathcal{O}_X$ splits.

Mixed characteristic: What to do?

Starting point:

Theorem (Hochster's direct summand conjecture (André))

Let R be a regular Noetherian local ring.
Then every finite extension $R \subseteq S$ splits as an R-module map.

Candidate for a mild singularity in mixed characteristic:

X is a *splinter* $\overset{def}{\iff}$ $\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits for every finite surjection $f\colon Y \to X$.

Caveaut: X characteristic 0, then splinter $=$ normal

More history:

- Breakthrough work of Ma-Schwede on mixed characteristic singularities after André.

- **Theorem (Bhatt!!!)**

  R Noetherian local domain of mixed characteristic.
  Then R^+ is Cohen-Macaulay up to p-completion

  Consequence: Kodaira vanishing up to finite covers.

Geometric applications

Theorem (Bhatt-Ma-Patakfalvi-Schwede-Tucker-Waldron-W., Takamatsu-Yoshikawa)

3-dimensional mixed characteristic Minimal Model Program over $\mathbb{Z}[1/30]$

Assuming four-dimensional resolutions:

Theorem (Hacon-W., Xie-Xue)

4-dimensional semistable Minimal Model Program over $\mathbb{Z}[1/30]$

Corollary (Hacon-W.)

Let X *and* Y *be birational Calabi-Yau 3-folds in characteristic* $p > 5$.

Then X *lifts to characteristic* $0 \implies Y$ *lifts to characteristic* 0.

Simplifying assumption: X is Gorenstein (K_X is Cartier and X is CM).

characteristic 0	characteristic p	mixed characteristic
rational	F-rational	splinter

Characteristic 0

X rational $\iff \pi_*\omega_Y = \omega_X$ for a resolution $\pi\colon Y \to X$

Mixed characteristics

$$X \text{ splinter} \iff f_*\omega_Y \twoheadrightarrow \omega_X \ \text{ for all } f\colon Y \to X \text{ finite surjection}$$

$$\overset{\mathit{Bhatt}}{\iff} \underbrace{g_*\omega_Y \twoheadrightarrow \omega_X \ \text{ for all } g\colon Y \to X \text{ alteration}}_{(\dagger)}$$

General definition

X is alteration-splinter $\overset{\mathit{def}}{\iff}$ $(\dagger)$ holds

Ideals measuring singularities

Goal 1: show that the locus of prime ideals p such that R_p is an alteration-splinter is open (cf. Datta-Tucker).

Goal 2: find an ideal $I \subseteq R$ such that

1. R_p is an alteration-splinter $\iff p \notin \mathrm{Supp}\, R/I$
2. I is calculated using alterations of R.

We are building on the work of Blickle-Schwede-Tucker in positive characteristic (test ideals via finite maps and alterations) which in turn builds on the work of Hochster, Huneke, Smith, ...

Ideal measuring rational singularities in characteristic 0:
(Grauert-Riemenschneider sheaf)

$$\mathcal{I}(\omega_X) \overset{def}{:=} \bigcap_{\pi\colon Y \to X} \pi_*\omega_Y \text{ over all projective birational } \pi\colon Y \to X$$
$$= \pi_*\omega_Y \text{ for a resolution } \pi\colon Y \to X$$

Properties:

- $\mathcal{I}(\omega_X) \subseteq \omega_X$ is a coherent ideal sheaf
- $\mathcal{I}(\omega_X) = \omega_X \iff X$ has rational singularities.

Ideal measuring alteration-splinters: (test ideal)

$$\tau(\omega_X) := \bigcap_{g\colon Y \to X} \mathrm{im}\,(g_*\omega_Y \to \omega_X) \text{ over all alterations } g\colon Y \to X$$

By definition, $\tau(\omega_X) = \omega_X \iff X$ is alteration-splinter.

Conjecture

$$\tau(\omega_X) = \mathrm{im}\,(g_*\omega_Y \to \omega_X) \textit{ for some alteration } g\colon Y \to X.$$

In particular, $\tau(\omega_X)$ *is coherent, i.e.* $\tau(\omega_R[\frac{1}{f}]) = \tau(\omega_R)[\frac{1}{f}]$.

Theorem (BMPSTWW II (in progress))

The above conjecture holds up to small p-perturbation.

Specifically, $\tau(\omega_X, p^{\epsilon})$ *is coherent and calculated by a single alteration.*

Applications

- openness of *almost*-splinter locus
- effective global generation, Briancon-Skoda, subadditivity, ...
- $\tau(\omega_X) = \tau_{\mathrm{HLS}}(\omega_X)$ (*ideal of Hacon-Lamarche-Schwede*)
- non-archimedean Calabi-Yau problem in mixed characteristics (*Fang-Gubler-Künnemann*)
- ...

Idea of the proof.

Goal: find an intrinsic definition of $\tau(\omega_X)$ using topological methods.

(motivated by work of Bhatt-Blickle-Lyubeznik-Singh-Zhang in positive characteristic)

Topological methods over $\mathbb{C}$:

Singular cohomology $H^i(X, \mathbb{C})$:

- good properties when X smooth
- many things break when X is singular (e.g. Poincaré duality)

Intersection cohomology (Goresky-MacPherson, Deligne)

There exists a complex $\mathrm{IC}_X \in D^b_{\mathrm{cons}}(X)$ such that

$$I^p H^i(X, \mathbb{C}) := \mathbb{H}^i(X, \mathrm{IC}_X[-d])$$

has many good properties (e.g. Poincare duality).

Remark

- $\mathrm{IC_X} = \mathbb{C}[d]$ if X smooth
- $\mathrm{IC_X}$ is an example of a perverse sheaf

$\mathrm{Perv}(X) \subseteq D^b_{\mathrm{cons}}(X)$ abelian category of *nice* complexes of cons. sheaves.

Intrinsic definition of $\mathcal{I}(\omega_X)$:

Fact

$$\mathcal{I}(\omega_X) = \mathrm{im}(H^0\mathrm{RH}^{\mathrm{Higgs}}(\mathrm{IC}_X) \to \omega_X)$$

Higgs Riemann-Hilbert ($= \mathrm{GR}_\bullet \mathrm{DR}(-)$):

$\mathrm{RH}^{\mathrm{Higgs}} \colon D^b_{\mathrm{HM}}(X) \to D^b_{\mathrm{coh}}(X)$.

Derived category of Hodge modules:

Intuition: $D^b_{\mathrm{HM}}(X) = D^b_{\mathrm{cons}}(X)$ but with some additional structure

Example: If X smooth, then

$$\mathrm{RH}^{\mathrm{Higgs}}(\mathbb{Q}) = \mathcal{O}_X[0] \oplus \Omega^1_X[-1] \oplus \cdots \oplus \Omega^d_X[-d]$$

New p-adic setting:

- $V = \mathbb{Z}_p$ and $V_\infty = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$.
- X_∞ proper flat scheme over V_∞.

Theorem (p-adic Riemann-Hilbert of Bhatt-Lurie)

There exists a functor $\mathrm{RH}^{\mathrm{Higgs}} \colon D^b_{\mathrm{cons}}(X_\infty[\frac{1}{p}], \mathbb{Z}_p) \to D^b_{\mathrm{coh}}(X_\infty)^a$ *such that*

- *it commutes with pushforward and duality*
- $\mathrm{RH}^{\mathrm{Higgs}}(\mathbb{Z}_p) = \mathcal{O}_{X_\infty,\mathrm{perfd}}$
- *it is left t-exact for perverse t-structures on both sides*

Remark:

For X smooth, $\mathcal{O}_{X,\mathrm{perfd}}[\frac{1}{p}] \simeq \mathcal{O}_{X[\frac{1}{p}]}[0] \oplus \Omega^1_{X[\frac{1}{p}]}[-1] \oplus \cdots \oplus \Omega^d_{X[\frac{1}{p}]}[-d]$.

The following theorem implies the main theorem.

Theorem (BMPSTWW II (in progress))

X – proper flat over V

$X_\infty := X \otimes_V V_\infty$

Then

(A) $\tau(\omega_{X_\infty}) = \underbrace{\mathrm{im}(H^0\mathrm{RH}^{\mathrm{Higgs}}(\mathrm{IC}_{X_\infty}) \to \omega_{X_\infty})}_{(\star)}$

(B) $\tau(\omega_X) = t(\tau(\omega_{X_\infty}))$ *for some special map* $t\colon \mathcal{O}_{X_\infty} \to \mathcal{O}_X$

For simplicity, we omit technical shifts of indices related to the fact that the perverse t-structures are relative.

Sketch of the proof of (A): After completion at $\mathfrak{m}$, by Matlis duality:

$$\begin{aligned}\tau(\omega_{X_\infty}) &= \operatorname{im}\left(H^d_{\mathfrak{m}}(\mathcal{O}_{X_\infty}) \to H^d_{\mathfrak{m}}(\pi_*\mathcal{O}^+_{X_\infty})\right)^\vee \\ (\star) &= \operatorname{im}\left(H^d_{\mathfrak{m}}(\mathcal{O}_{X_\infty}) \to H^d_{\mathfrak{m}}(\mathrm{RH}^{\mathrm{Higgs}}(\mathrm{IC}_{X_\infty}[-d]))\right)^\vee,\end{aligned}$$

where $\pi\colon X^+_\infty \to X_\infty$.

By Bhatt, $\mathrm{IC}_{X_\infty} \to \pi_*(\mathbb{Z}_p[d])_{X^+_\infty}$ is an injection of perverse sheaves on $X_\infty[\frac{1}{p}]$.

Apply RH: $\mathrm{RH}^{\mathrm{Higgs}}(\mathrm{IC}_{X_\infty}[-d]) \to \mathrm{RH}^{\mathrm{Higgs}}(\pi_*(\mathbb{Z}_p)_{X^+_\infty}) = \pi_*\mathcal{O}^+_{X_\infty}$.

By perverse left t-exactness: $H^d_{\mathfrak{m}}(\mathrm{RH}^{\mathrm{Higgs}}(\mathrm{IC}_{X_\infty}[-d])) \to H^d_{\mathfrak{m}}(\pi_*\mathcal{O}^+_{X_\infty})$ is injective.

Hence, $\tau(\omega_{X_\infty}) = (\star)$.