Perfectoid signature and an application to étale fundamental groups

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Detecting singularities

Let $(R, \mathfrak{m})$ be a Noetherian local ring.

Theorem (Kunz)

In char. $p > 0$, F *Frobenius*

R *is regular*

$\Leftrightarrow F_* R = R^{1/p}$ *fflat R-module*

$\Leftrightarrow R_{\text{perf}} = \operatorname{colim}_e F^e_* R$ *fflat R-module.*

Detecting singularities part 2

Since mixed characteristic conference:

Theorem (Bhatt-Iyengar-Ma)

In mixed characteristic $(0, p)$*,*
R is regular
$\Leftrightarrow$ *there is $R \to B$ with B perfectoid, fflat / R.*
(also an almost fflat version)

Instead of detecting sings, measure them.

Measuring singularities in char p

Suppose $(R, \mathfrak{m}, k = k^p)$ is complete Noetherian local domain characteristic $p > 0$, $\dim R = d$.

If R is regular, $F^e_* R = R^{1/p^e}$ is free over R of rank p^{ed}.

$$\#\text{gens } R^{1/p^e} = \text{length}(R^{1/p^e} / \mathfrak{m} R^{1/p^e}) = p^{ed}.$$

Do an example! (on a board)

If R^{1/p^e} is not free, then since it is free of rank p^{ed} at generic point, we see:

$$\#\text{gens } R^{1/p^e} = \text{length}(R^{1/p^e} / \mathfrak{m} R^{1/p^e}) > p^{ed}.$$

Hilbert-Kunz multiplicity

Definition (Kunz, Monsky)

If $J \subseteq R$ is $\mathfrak{m}$-primary, *Hilbert-Kunz mulitiplicity*

$$e_{HK}(J, R) = \lim_{e \to \infty} \frac{\text{length}(R^{1/p^e}/JR^{1/p^e})}{p^{ed}}.$$

If R is regular & $J = \mathfrak{m}$, $e_{HK}(R) := e_{HK}(\mathfrak{m}, R) = 1$.

If R *not* regular & $J = \mathfrak{m}$, $e_{HK}(R) > 1$. (Watanabe-Yoshida)

Bigger $e_{HK}(R)$ means *MORE* singular R.

$e_{HK}(R)$ can be irrational (Brenner).

Measuring singularities in char p part 2

Suppose $(R, \mathfrak{m}, k = k^p)$ is complete Noetherian local domain characteristic $p > 0$, $\dim R = d$.

Write

$$R^{1/p^e} = R^{\oplus a_e} \oplus M_e \qquad M_e \text{ no free } R\text{-summands}$$

$$a_e = \operatorname{length} R/I_e,$$

$$I_e := \{x \in R \mid R \xrightarrow{1 \mapsto x^{1/p^e}} R^{1/p^e} \text{ does not split}\}$$

(Sketch why, on a board!)

You can ask what percentage of R^{1/p^e} is free, asymptotically.

Definition (Smith-Van den Bergh, Huneke-Leuschke, Tucker)

F-signature is defined

$$s(R) := \lim_{e\to\infty} \frac{a_e}{p^{ed}}.$$

The "fraction" of R^{1/p^e} that is *free*, asymptotically.

$0 \leq s(R) \leq 1$.

Smaller $s(R)$ means *MORE* singular R.

R regular $\Leftrightarrow$ $s(R) = 1$ (Watanabe-Yoshida)

Examples

Example (ADE surface singularities)

For surface singularities, $p > 5$.

type	equation	$s(R)$	$e_{HK}(R)$
(A_n)	$xy + z^{n+1}$	$\frac{1}{n+1}$	$2 - s(R)$
(D_n)	$x^2 + yz^2 + y^{n-1}$	$\frac{1}{4(n-2)}$	$2 - s(R)$
(E_6)	$x^2 + y^3 + z^4$	$\frac{1}{24}$	$2 - s(R)$
(E_7)	$x^2 + y^3 + yz^3$	$\frac{1}{48}$	$2 - s(R)$
(E_8)	$x^2 + y^3 + z^5$	$\frac{1}{120}$	$2 - s(R)$

$e_{HK} = 2 - s(R)$ since mult. 2 hypersurface (only mult. 2)

A more *interesting* example

Example (Monsky)

$k = \overline{k}$ char 2.

$$R = k[x,y,z]/((\lambda^2+\lambda)x^2y^2 + z^4 + xyz^2 + (x^3+y^3)z)$$

then $e_{HK}(R) = 3 + 4^{-m}$ where $m = [\mathbb{F}_2(\lambda) : \mathbb{F}_2]$.
If λ transcendental, $e_{HK}(R) = 3$. Otherwise it's > 3 (depends on degree of $\lambda/\mathbb{F}_p$).

Ie, this is a family over $\operatorname{Spec} k[\lambda]$. Then very general fiber is least singular. Lack of semi-continuity.

Positive signature and motivation

Theorem (Aberbach-Leuschke)

$s(R) > 0 \Leftrightarrow R$ is strongly F-regular.

- Whatever strong F-regularity is, you can define it as above.
- strong F-regular is char. $p > 0$ analog of klt sings/$\mathbb{C}$.
- Expect $s(R) > 0$ means R behaves like klt sings.

There was conj. (Kollár), $|\pi_1(\partial B \cap X_{\mathrm{nonsing}})| < \infty$, B small ball around klt singularity $x \in X/\mathbb{C}$.

Theorem (Braun, Xu-Zhuang, *cf.* Xu, Bhatt-Gabber-Olsson)

$|\pi_1(\partial B \cap X_{\mathrm{nonsing}})| \leq d^d / \widehat{\mathrm{vol}}(x, X)$ *(norm. vol., cf. Li-Liu-Xu, etc)*

An application

- If $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, l)$ finite split étale-in-codim.=1, then:

$$s(R) \cdot [K(S) : K(R)] = s(S) \cdot [l : k].$$

(Say a word about the proof, on a board!)

Corollary (Carvajal-Rojas - S. - Tucker)

If $(R, \mathfrak{m}, k = \overline{k})$ is strongly F-regular, then

$$|\pi_1^{ét}((\operatorname{Spec} \widehat{R})_{\text{nonsing}})| \leq 1/s(R).$$

Mixed characteristic

Our goal, $(R = \widehat{R}, \mathfrak{m}, k = k^p)$ mixed characteristic.

- Find analog of $e_{HK}(J, R)$.
- Find analog of $s(R)$.
- Prove analogous results from char $p > 0$.
- Conclude étale fun. group. $(k = \overline{k})$ for "nice" R

$$|\pi_1^{\text{ét}}((\operatorname{Spec} R)_{\text{nonsing}})| \leq 1/s(R) < \infty.$$

No Frobenius! (or resolution of singularities).

An idea! Instead of R^{1/p^e}, maybe we can use R_{perf} in char. $p > 0$.

.... then in mixed char. use perfectoidization (Bhatt-Scholze).

Normalized length

Recall normalized length (Faltings, *cf.* Gabber-Ramero).

$A = k[\![x_1, \dots, x_n]\!]$ OR $A = W(k)[\![p = x_1, x_2, \dots, x_n]\!]$ Consider:
$A \to A_e := A[x_1^{1/p^e}, \dots, x_d^{1/p^e}]$ ($e = \infty$ ok, but p-complete).

(Write A_∞ defn on board, completed)

- M is an A_∞-mod., $\mathfrak{m}^N \cdot M = 0$. Define $\lambda_\infty(M)$.
- If M'' f.p. $M'' = M''_e \otimes_{A_e} A_\infty$,

$$\lambda_\infty(M'') = \mathrm{length}(M''_e)/p^{ed}.$$

- If M' f.g, $\lambda_\infty(M') = \inf_{M'' \twoheadrightarrow M'} \lambda_\infty(M'')$.
- In general $\lambda_\infty(M) = \sup_{M' \hookrightarrow M} \lambda_\infty(M')$.

Using normalized length

Theorem (Cai-Lee-Ma-S.-Tucker)

If $(R, \mathfrak{m}, k = k^p)$ *complete Noeth. local domain char.* $p > 0$

Fix $A \subseteq R$ *Noether norm. (Cohen). Then*

$$e_{HK}(J, R) = \lambda_\infty(R_{\text{perf}}/JR_{\text{perf}}) \quad \textit{and}$$

$$s(R) = \lambda_\infty(R_{\text{perf}}/I_\infty)$$

where $I_\infty = \{x \in R_{\text{perf}} \mid R \xrightarrow{1 \mapsto x} R_{\text{perf}} \textit{ not split}\}$.

Note $I_\infty = \bigcup_e I_e^{1/p^e}$.

Mixed characteristic definition

$(R, \mathfrak{m}, k = k^p)$ mixed char. complete Noetherian domain.

- Fix $A := W(k)[\![x_2, \ldots, x_d]\!] \subseteq R$.
- Let $R^X_{\mathrm{perfd}} := (R \otimes_A A_\infty)_{\mathrm{perfd}}$ (Bhatt-Scholze).
- Note R^X_{perfd} is a ring, an A_∞-module. Can take normalized length of A_∞-mods as before.

Definition

Perfectoid Hilbert-Kunz: $e^X_{\mathrm{perfd}}(J, R) := \lambda_\infty(R^X_{\mathrm{perfd}} / JR^X_{\mathrm{perfd}})$.

Perfectoid signature: $s^X_{\mathrm{perfd}}(R) := \lambda_\infty(R^X_{\mathrm{perfd}} / I_\infty)$,

where $I_\infty = \{x \in R^X_{\mathrm{perfd}} \mid R \xrightarrow{1 \mapsto x} R^X_{\mathrm{perfd}}$ *not* split$\}$.

Agrees with char. $p > 0$ definitions.
(Write on board!)

Properties of perfectoid Hilbert-Kunz

Theorem (CLM_T)

- $e^{x}_{\text{perfd}}(R) := e^{x}_{\text{perfd}}(\mathfrak{m}, R) \geq 1$.
- $e_{\text{perfd}}(R) = 1 \;\Leftrightarrow\;$ *R is regular.*
- *If* $J = (f_1, \ldots, f_d)$, $\sqrt{J} = \mathfrak{m}$ *(param. ideal), then* $e^{x}_{\text{perfd}}(J) = e(J, R)$ *(Hilbert-Samuel multiplicity).*
- *If* $I \subseteq J$ $\mathfrak{m}$*-primary then*

$$e^{x}_{\text{perfd}}(I, R) = e^{x}_{\text{perfd}}(J, R) \quad \Leftrightarrow \quad IB = JB$$

for some perfectoid BCM (Big Cohen-Macaulay) B.
(Exist by André, Gabber. Note $\widehat{R^+}$ *is one such by Bhatt.)*

$e_{\text{perfd}}(R)$ bigger means *R is more singular.*

Questions

Questions

- independent of $A \subseteq R$, $\underline{x} = x_2, \ldots x_d$?
- semi-continuity Zariski topology? Even:

$$e_{\text{perfd}}(R) \geq e_{\text{perfd}}(\widehat{R_Q})$$

$$s_{\text{perfd}}(R) \leq s_{\text{perfd}}(\widehat{R_Q}).$$

- fflat ascent? (Lech)

$$(R, \mathfrak{m}) \xrightarrow{\text{fflat}} (S, \mathfrak{n})$$

$$e_{\text{perfd}}(R) \leq e_{\text{perfd}}(S)?$$

“Nice” rings, BCM-regularity

Definition

R is *weakly BCM-regular* if $R \hookrightarrow B$ splits/pure for every perfectoid BCM (Big Cohen-Macaulay) B.

R is *BCM-regular* if it is also $\mathbb{Q}$-Gorenstein (ie, hypersurface)
For pair $(R, \Delta \geq 0)$ *BCM-regular* makes sense (log $\mathbb{Q}$-Gor.).

(BCM-regular $\Rightarrow$ KLT, = strongly F-regular in char $p > 0$).

Example: log regular $\Rightarrow$ (R, Δ) BCM-regular for some Δ.

Example: $\mathbb{Z}_p[\![x, y, z]\!]/(p^3 + x^3 + y^3 + z^3)$ BCM-regular ($p > 3$).

Example: $\mathbb{Z}_p[\![y, z]\!]/(p^2 z - x(x - z)(x + z))$ *not* BCM-regular.

Properties of perfectoid signature

Detects regularity.

Theorem (CLM_T)

$0 \leq s^{X}_{\text{perfd}}(R) \leq 1.$

R is regular $\Leftrightarrow$ $s^{X}_{\text{perfd}}(R) = 1$

Detects BCM-regularity.

Theorem (CLM_T)

If $s^{X}_{\text{perfd}}(R) > 0$*, then R weakly BCM-regular.*

If R (or (R, Δ)*) is BCM-regular, then* $s^{X}_{\text{perfd}}(R) > 0$.

($\mathbb{Q}$*-gor.* $\Rightarrow$ *equiv.)*

Transformation rules

Suppose $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, l)$ finite, étale in codim 1.

Theorem (CLM_T)

$$s^{\underline{x}}_{\text{perfd}}(R) \cdot [K(S) : K(R)] = s^{\underline{x}}_{\text{perfd}}(S) \cdot [l : k].$$

Transformation rule also works for μ_n-in-1-covers even if $p|n$ (for good choice of A, $\underline{x}$).

Proof idea.

- assume $k = l$.
- Extension is gen. free rank $[K(S) : K(R)] := r$.
- $S^{\underline{x}}_{\text{perfd}}/I^R_\infty S^{\underline{x}}_{\text{perfd}} \overset{g-\text{almost}}{\simeq} S^{\underline{x}}_{\text{perfd}}/I^S_\infty$. So have same normalized length.
- Want to compare $M = S^{\underline{x}}_{\text{perfd}}(R)/I^R_\infty S^{\underline{x}}_{\text{perfd}}$ and $N = \bigoplus^r R^{\underline{x}}_{\text{perfd}}/I^R_\infty$. Work mod p. $M \to N \to M$.

□

An example

As a consequence:

Example (ADE surface singularities)

S regular dim 3., $p > 5$.
$R = S/f$, $\mathfrak{m}_S = (x, y, z)$.

type	f	$s^x_{\mathrm{perfd}}(R)$	$e^x_{\mathrm{perfd}}(R)$
(A_n)	$xy + z^{n+1}$	$\frac{1}{n+1}$	$2 - s(R)$
(D_n)	$x^2 + yz^2 + y^{n-1}$	$\frac{1}{4(n-2)}$	$2 - s(R)$
(E_6)	$x^2 + y^3 + z^4$	$\frac{1}{24}$	$2 - s(R)$
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$e_{HK} = 2 - s(R)$ since mult. 2 hypersurface (only mult. 2)

For some cases, need careful choice $A \subseteq R$.
(Uses Carvajal-Rojas - Ma - Polstra - S. - Tucker).

Application

Theorem (CLM_T)

Suppose $(R, \mathfrak{m}, k = \overline{k})$ *complete Noeth. local. Set* $U = (\mathrm{Spec}\, R)_{\mathrm{nonsing}}$. *Then*

$$|\pi_1^{ét}(U)| \leq 1/s_{\mathrm{perfd}}^{x}(R).$$

In particular, if R is BCM-regular, then it's finite.

Furthermore, for careful choice of A,

$$|Cl(R)_{\mathrm{tors}}| \leq 1/s_{\mathrm{perfd}}^{x}(R).$$

Hence, R is BCM-regular $\Rightarrow$ *torsion in class group is finite.*

Other variants, like Greb-Kebekus-Peternell / $\mathbb{C}$ also work.

Thank you for listening!

Thank Bhargav for math we used!