# Diffracting prisms: resolution of a quandary

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# Outline

- Theme and variations
- Actions on envelopes
- Actions on cohomology and on categories of crystals
- Quandaries and their resolutions

# Thanks

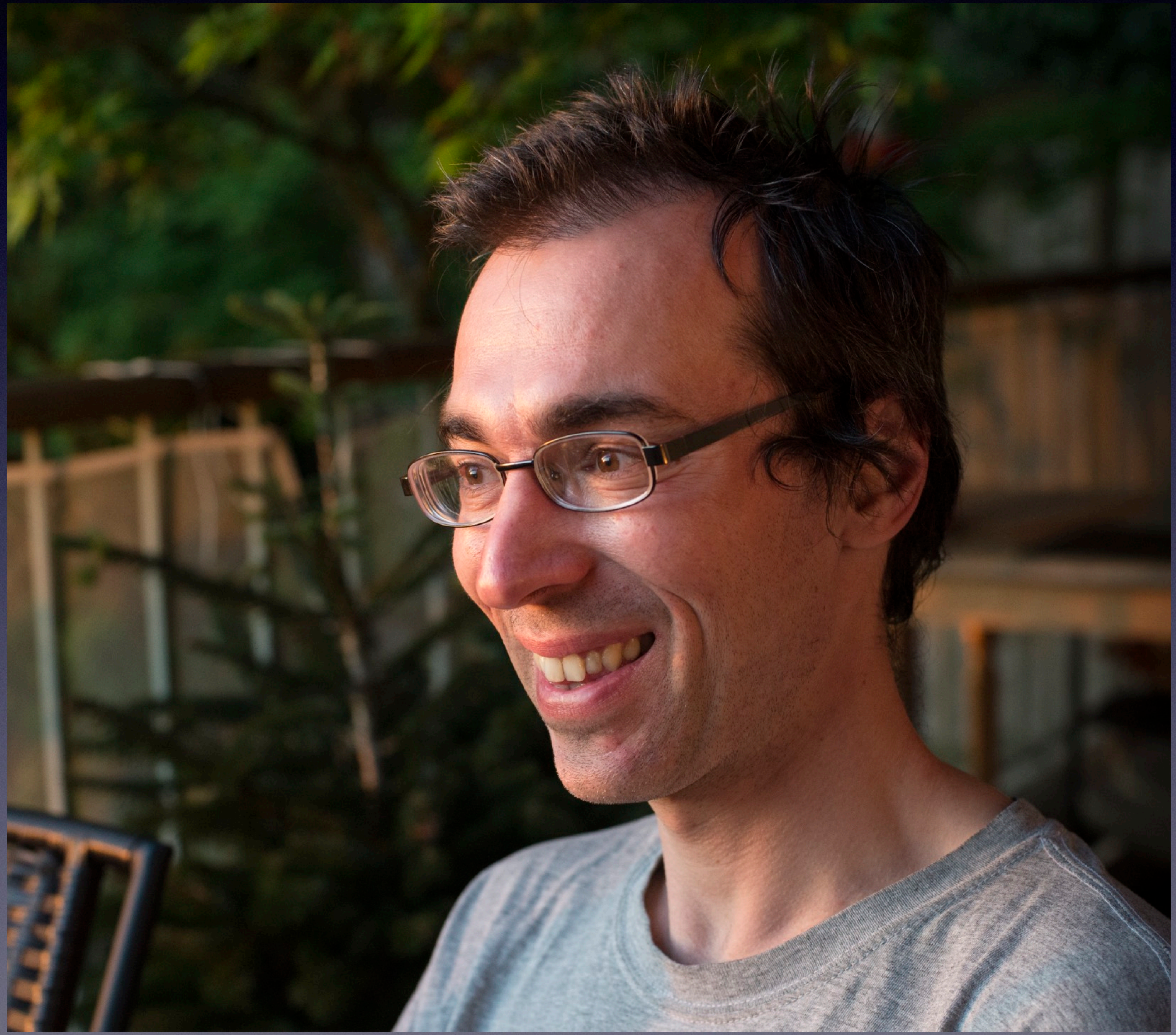
- Organizers
  - For inviting me to speak here, and in particular:
  - For giving me a chance to thank everyone.

# Luc Illusie

- First to tell me about new developments, constant companion.
- On hearing my idea about diffracting prisms:
  - “Your idea will never work.”
  - “I do not have the motivation to study your approach.”
  - “In your next lecture, be sure to explain that I do not approve.”
  - His opinion was justified, as you will see.

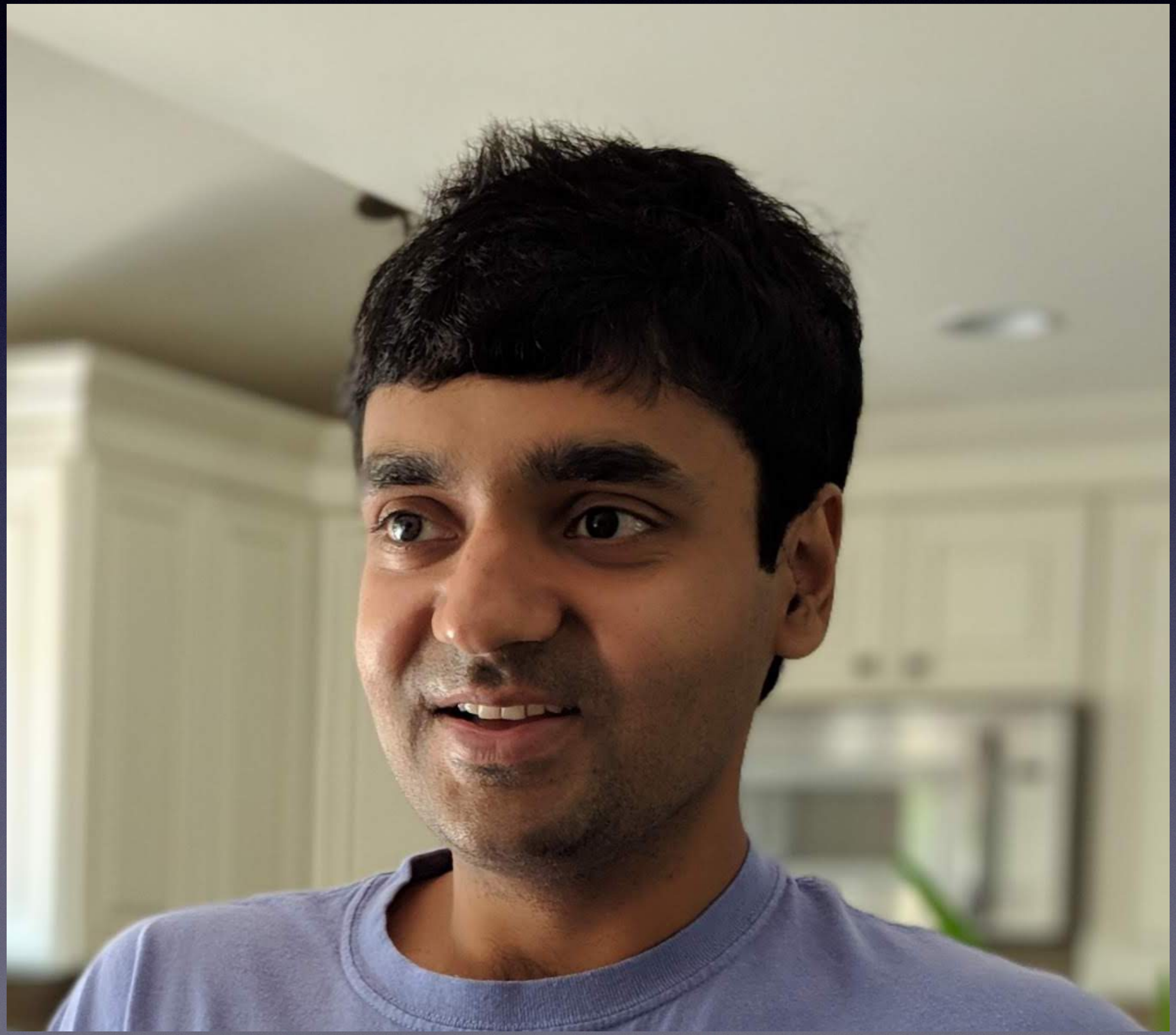
# Vadim Vologodsky

- Consultation and inspiration, past and present:
  - Joint work on the Cartier transform
  - Categorification of Drinfeld's idea
  - Patient explanations of his work and of that of Drinfeld and Bhatt-Lurie.
- On hearing my idea about  diffracting prisms:
  - "It would be great if your idea  works."
  - "I agree. Here is a general conjecture."



# Bhargav Bhatt

- Inspiration for all my work over the last five years:
- New construction of the DR-Witt complex
- Prisms
    - 2019 : ``You will like this:”
      Prismatic cohomology and Higgs fields
- On hearing my idea about diffracting prisms:
    - 2023 “To reconcile our different approaches, come to Princeton.”
    - “I am writing to share….another construction of the Sen operator….motivated by Arthur's idea of using a flat connection…, but wanted to do it intrinsically….”



# Theme and Variations

Hodge theory as $\mathbf{G}_m$ actions

# Classical Case:

- $X/\mathrm{C}$ smooth, projective
  - Hodge: $R\Gamma(\Omega^{\bullet}_{X/\mathrm{C}}, d) \sim R\Gamma(\bigoplus \Omega^{a}_{X/\mathrm{C}}[-a])$
    - $u \in \mathrm{G_m}$ acts on $R\Gamma(\Omega^{a}_{X})$ : multiplication by $u^{a}$.
  - Simpson:
    - $\mathrm{MIC_{ss}}(X/\mathrm{C}) \equiv \mathrm{HIG_{ss}}(X/\mathrm{C})$
    - Hence: $\mathrm{G_m}$-action on these categories.

# Recall:

- $\nabla : E \to \Omega^1_{Y/S} \otimes E$ is
  - a connection if $\nabla(fe) = df \otimes e + f\nabla(e)$
  - a $\lambda$-connection if $\nabla(fe) = \lambda df \otimes e + f\nabla(e)$
  - a Higgs field if $\nabla(fe) = f\nabla(e)$ (and $\nabla^2 = 0$)
    - equivalent: $S^{\bullet}T_{Y/S}$ - module structure on $E$.

# Early $p$-adic and char. $p$ results

- $X/k$ smooth, $k$ perfect of char. $p$, $S := Spf(W)$ , $\overline{S} := Spec(k)$
  - $F_{X/\overline{S}}\colon X \to X^{(p)}$ the relative Frobenius morphism.
- Deligne-Illusie:
  A lifting of $X$ to $W_2$ induces an isomorphism:
  - $F_{X/S*}(\tau_{<p}\Omega^\bullet_{X/k}) \sim \bigoplus_{a<p} \Omega^a_{X^{(p)}/k}[-a]$ (in $\mathrm{D}_{\mathrm{coh}}(O_{X^{(p)}})$).
  - Better: $\sim \bigoplus_{a<p} \mathscr{H}^a(\Omega^\bullet_{X/S})[-a]$
  - $\Omega^\bullet_{X/k} \sim \bigoplus_a \Omega^a_{X^{(p)}/k}[-a]$ (in $\mathrm{D}_{\mathrm{coh}}(O_{X^{(p)}})$). (Given also a lifting of $F_{X/\overline{S}}$.)

- Cartier transform (O. - Vologodsky):
  - $\mathrm{MIC}^{<\mathrm{p}}_{\mathrm{qn}}(X/\overline{S}) \equiv \mathrm{HIG}^{<\mathrm{p}}_{\mathrm{qn}}(X^{(p)}/\overline{S})$, or, better:
  - $\mathrm{MIC}^{\gamma}_{\mathrm{qn}}(X/\overline{S}) \equiv \mathrm{HIG}^{\gamma}_{\mathrm{qn}}(X^{(p)}/\overline{S})$
  - Hence an action of $\mathbf{G}_{\mathrm{m}}$ on both categories.

- F-transform (Faltings, Xu, Shiho):
- Given $Y/S$, with a lifting $\phi_Y$ of $F_{\overline{Y}}$,
  there is an equivalence of categories:
- $\mathrm{MICP}_{qn}(Y^{(p)}/S) \equiv \mathrm{MIC}_{qn}(Y/S)$: ($p$-connections to connections)
  - $(E', \nabla') \mapsto (E, \nabla)$, $E := \phi^*_{Y/S}(E)$,
    $\nabla(1 \otimes e') = p^{-1}(\mathrm{id_E} \otimes \phi^*_{\mathrm{Y/S}})(\nabla'(\mathrm{e}))$
  - Mod $p$: $\mathrm{HIG_{qn}}(\overline{Y}^{(p)}/\overline{S}) \equiv \mathrm{MIC_{qn}}(\overline{Y}/\overline{S})$
  - Compatible with cohomology.

# New prismatic approach

- $\mathrm{G}_{\mathrm{m}}^{\gamma} :=$ the divided power envelope of the identity in $\mathrm{G}_{\mathrm{m}}$.
- $1 \to \mu_p \to \mathrm{G}_{\mathrm{m}}^{\gamma} \to \mathrm{G}_{\mathrm{a}}^{\gamma} \to 0$ : exact, split in char. $p$.
- Get: a $\mathrm{G}_{\mathrm{m}}^{\gamma}$-action on ***all*** of $\Omega^{\bullet}_{X/k}$ (in derived category). (Drinfeld)
- a $\mathrm{G}_{\mathrm{m}}^{\gamma}$-action on $\mathrm{MIC}_{qn}(X/k)$. ( Vologodsky)
- Bhatt-Lurie: ``The diffracted Hodge complex"

# My goals today

- $\mathbb{G}_m^{\gamma}$-action on:
  - Prismatic envelopes
  - Prismatic cohomology complexes
  - Categories of crystals
- Explanation of quandaries encountered

# Actions on envelopes

# The setup

- Notation: If $T$ is a formal scheme, $\overline{T}$ is its reduction mod $p$.
- $X/\overline{S}$ smooth, $Y/S$ smooth ($p$-completely)
- $F_X\colon X \to X$, Frobenius, $\phi_Y\colon Y \to Y$, lift of $F_{\overline{Y}}$
- $\phi_Y^*\colon O_Y \to O_Y : \phi_Y^*(f) = f^p + p\delta(f)$
- $(Y, \phi_Y)$ is a $p$-torsion free (formal) $\delta$-scheme.

# $\delta$-schemes

- Def: A "$\delta$-scheme" is a $p$-adic formal scheme $Z$ endowed with a map $\delta\colon\, O_Z \to O_Z$ such that
  - $\delta(1) = 0$
  - $\delta(f+g) = \delta(f) + \delta(g) - p^{-1}\sum_{1}^{p-1}\binom{p}{i} f^i g^{p-i}$
- $\delta(fg) = \delta(f)g^p + f^p\delta(g) + p\delta(f)\delta(g)$
- So $\phi(f) := f^p + p\delta(f)$ defines a Frobenius lift

- We have categories and functors:
  - Forget: $: Sch_\delta \to Sch_p$
  - Left adjoint: $\mathrm{W}: Sch_p \to Sch_\delta$ (Joyal)
  - Right adjoint: $\mathrm{J}: Sch_p \to Sch_\delta$ (Buium, Borger)
- Thus, for $T \in Sch_p$, there are universal:
  - $i: T \to \mathrm{W}(T)$
  - $\pi: \mathrm{J}(T) \to T$
- Formation of $\mathrm{W}(T)$ and of $\mathrm{J}(T)$ is compatible with étale localization.

# Envelopes

- $X \subseteq \overline{Y} \subseteq Y$ ($\delta$-scheme)
- $\Delta_X(Y) \to Y$ is the universal $\delta$-map from a ($p$-torsion free) $\delta$-scheme to $Y$ such that $\overline{\Delta}_X(Y) \to \overline{Y}$ factors through $X$.
- $D_X(Y) \to Y$ is the universal map from a $p$-torsion free formal scheme to $Y$ such that $\overline{D}_X(Y) \to \overline{Y}$ factors through $X$.
  - (Used by Oyama and Xu to study the Cartier transform.)
- Have $\Delta_X(Y) \xrightarrow{h} D_X(Y) \to Y$.

# Important special case

- $i_{Y/Z}\colon Y \to Z$, a closed immersion of smooth $\delta$-schemes over $S$.
- Get sections: $i_{Y/S}\colon Y \to D_Y(Z)$, $i_{Y/\Delta}\colon Y \to \Delta_Y(Z)$
- Theorem: $i_{Y/\Delta}\colon Y \to \Delta_Y(Z)$ is a PD-immersion.
    - In fact, $\Delta_Y(Z)$ is the PD-envelope of $i_{Y/S}\colon Y \to D_Y(Z)$.

# Dilatations and Dilations

- $D_X(Y) \to Y$ is the "dilatation" of $X$ in $Y$.
- (The affine piece $D^+(p)$ of the blow-up of $X$ in $Y$)
- Have $\rho_D : I_{X/Y} \to O_D : \rho_D(p) = 1$.

- When $X \to Y$ is a ``d-regular'' embedding, $D_X(Y)$ is also the "dilation" of $X$ in $Y$:
(use a theorem of C. Huneke: $S^n I_{X/Y} \cong I^n_{X/Y}$).

- Thus, $\rho_D : I_{X/Y} \to O_D$ is universal:

- For all $\pi_T : T \to Y$,

  - $\{T \to D_X(Y)\} = \{\rho_T : \pi_T^*(I_{X/Y}) \to O_T : \rho_T(p) = 1\}$.

  - Note: makes sense for quasi-ideals too.

- Theorem: $X \to Y$ a regular immersion also implies:
  - $\Delta_X(Y) = \mathrm{J}(D_X(Y))$
- Thus, if $T \in Sch_p$ and $\pi_T\colon\ T \to Y$ is given, we get:
  - $\mathrm{W}(T) \to Y$, morphism of $\delta$-schemes.
  - $$\begin{aligned} Mor(T, \Delta_X(Y)) &\cong Mor_\delta(\mathrm{W}(T), \Delta_X(Y)) \\ &= Mor_\delta(\mathrm{W}(T), \mathrm{J}(D_X(Y)) \end{aligned}$$
  - $\cong Mor(\mathrm{W}(T), D_X(Y)) \cong \{\rho\colon I_{X/Y} \to O_{\mathrm{W}(T)} : \rho(p) = 1\}$

# Actions on dilations

- Theorem: $D_X(Y)$ has an action of $N_{X/\overline{Y}}$, making it a pseudo-torsor over $Y$.
- Re: $D_X(Y)(T) = \{\rho_T \colon I_{X/Y} \to \pi_{T*}(O_T) : \rho_T(p) = 1\}$
- Clearly a pseudo-torsor under $\{s \colon I_{X/Y} \to \pi_{T*}(O_T) : s(p) = 0\}$ .
- $D_X(Y)(T) \neq \emptyset$ implies $\pi_T^\sharp(I_{X/Y}) \subseteq pO_T$, which implies:
- $\{s \colon I_{X/Y} \to \pi_{T*}(O_T) : s(p) = 0\} = \{\bar{s} \colon I_{X/\overline{Y}}/I^2_{X/\overline{Y}} \to O_T\} \ = N_{X/\overline{Y}}(T)$.

- A lifting of $X$ in $Y$ defines a section of $\overline{D}_X(Y) \to X$. (mod $p^2$ is enough).

- Hence an isomorphism: $\overline{D}_X(Y) \cong N_{X/\overline{Y}}$

- Hence an action of $\mathrm{G_m}$ on $\overline{\mathrm{D}}_X(Y)$.

- My idea: the "restriction" of this action to $\mathrm{G}_\mathrm{m}^\gamma$ lifts to $\overline{\Delta}_X(Y)$, using "parallel transport."

- Need: Connection on $\overline{\Delta}_X(Y)$, viewed over $\overline{D}_X(Y)$.

# The quandary

- Calculated an example of envelope and cohomology complex.
- Saw that ***no*** compatible action was possible.
- Went ahead anyway.

# The connection on $\overline{\Delta}_X(Y)$

- Note: $\Omega^1_{D_X(Y)/Y} \cong \Omega^1_{\overline{D}_X(Y)/X} \cong \pi^*(I_{X/\overline{Y}}/I^2_{X/\overline{Y}})$
- Theorem: There is a unique (integrable, q-nilpotent) connection: $\nabla : O_{\overline{\Delta}_X(Y)} \to O_{\overline{\Delta}_X(Y)} \otimes \Omega^1_{D_X(Y)/Y}$ such that
  - $\nabla(f) = df$ if $f \in h^*(O_{\overline{D}_X(Y)})$
  - $\nabla(fg) = f\nabla(g) + g\nabla(f)$ if $f, g \in O_{\overline{\Delta}_X(Y)}$
  - $\nabla(\bar{\delta}(f)) = -\bar{f}^{p-1}\nabla\bar{f}$ if $f \in O_{\Delta_X(Y)}$

- The connection $\nabla$ produces a stratification $\epsilon_\Delta$, which we reinterpret as follows.
- $\overline{D}_X(Y) \times_X N_{X/\overline{Y}} \cong \overline{D}_X(Y) \times_X \overline{D}_X(Y) : (z, \eta) \mapsto (z, z\eta)$
- $\overline{D}_X(Y) \times_X N^\gamma_{X/\overline{Y}} \cong \overline{D}_X(Y) \times^\gamma_X \overline{D}_X(Y)$
- So a PD-stratification $\epsilon_\Delta$ on $\overline{\Delta}_X(Y)$ is an $N^\gamma_{X/\overline{Y}}$-action

# Action on $\Delta_X(Y)$

- Theorem: (Bhatt, or Bhatt-Lurie?)
- The action of $N_{X/\overline{Y}}$ on $\overline{D}_X(Y)$ lifts to an action of $N^{\gamma}_{X/\overline{Y}}$ on $\overline{\Delta}_X(Y)$.
  - and $\overline{\Delta}_X(Y)$ becomes a torsor for this action.
- Turns out: This coincides with my construction!

- Re: Given $T \to \overline{Y} \to Y$, $\{T \to \Delta_X(Y)\} \cong \{\mathrm{W}(T) \to D_X(Y)\}$
  - Torsor under
    - $\{\tilde{s}: I_{X/\overline{Y}}/I^2_{X/\overline{Y}} \to O_{\mathrm{W}(T)}\} = N_{X/\overline{Y}}(\mathrm{W}(T)) \cong$
    - $\{s_\Gamma: \Gamma^\bullet(I_{X/\overline{Y}}/I^2_{X/\overline{Y}}) \to O_T\} = N^\gamma_{X/\overline{Y}}(T)$
    - $\tilde{s}(x) = (s_\Gamma(x), -s_\Gamma(x^{[p]}), s_\Gamma(x^{[p^2]}), \cdots)$ (formula of Drinfeld).

# The $\mathrm{G}_{\mathrm{m}}^{\gamma}$-action on $\overline{\Delta}_X(Y)$

- Recall: a lifting $\tilde{X} \to Y$ induces an action of $\mathrm{G}_{\mathrm{m}}$ on $\overline{D}_X(Y)$:
  - $r_D \colon \overline{D}_X(Y) \times \mathrm{G}_{\mathrm{m}} \to \overline{D}_X(Y)$
- The stratification $\epsilon_\Delta$ of $\overline{\Delta}_X(Y)$ lifts
  - PD-paths in $\overline{D}_X(Y)$ to PD-paths in $\overline{\Delta}_X(Y)$.

# Parallel transport lifts the $\mathbf{G}^{\gamma}_{\mathbf{m}}$-action

$\epsilon_{\delta}$

$z \in \overline{\Delta}_X(Y)$

$z$ $z\lambda$

$\lambda \in \mathrm{G}^{\gamma}_{\mathrm{m}}$

$\overline{h}$ $\downarrow$

$\tilde{r}^{\gamma}_{D}$

$\overline{D}_X(Y)$

$\overline{h}(z)$ $h(z)\lambda$

# The Sen operator

- Derivative of the action with respect to $ud/du$, evaluated at $u = 1$
- Get vector fields $\theta_D$ and $\theta_\Delta$ on $\overline{D}_X(Y)$ and $\overline{\Delta}_X(Y)$, respectively.
- $\theta_D$ is the Euler vector field on $N_{X/\overline{Y}}$:
- $\theta_D = \sum s_i d/ds_i$ in coordinates
- $\theta_\Delta = \nabla_{\theta_D}$.

# Local formulas for the action

- Suppose $\tilde{X} \subseteq Y$ is a smooth lift of $X$ and $x \in I_{\tilde{X}/Y}$. We have:
- $\overline{\rho}_D(x) \in O_{\overline{D}_X(Y)}$,
- $\overline{\rho}_\Delta(x) \in O_{\overline{\Delta}_X(Y)}$,
- $\overline{\delta}(\rho_\Delta(x)) \in O_{\overline{\Delta}_X(Y)}$.

# Formulas:

- $r_D^*(\bar{\rho}_D(x)) = u\bar{\rho}_D(x)$
- $r_\Delta^*(\bar{\delta}(\rho_\Delta(x))) \ = \bar{\delta}(\rho_\Delta(x)) + \log(u)\bar{\pi}_\Delta^*(\bar{\delta}(x))$
- Thus, $\bar{\delta}(\rho_\Delta(x))$ is fixed iff $\bar{\pi}_\Delta^*(\bar{\delta}(x)) = 0$.

- In fact, the following are equivalent:
  - $\overline{\delta}(\rho_\Delta(x))$ is fixed under the action
  - $\delta(x) \in I_{X/Y}$
  - $\rho_\Delta(x)^{[n]} \in O_{\Delta_X(Y)}$ for all $n$.
- They imply: $r^*_\Delta(\overline{\rho}_\Delta(x)^{[p]}) = \overline{\rho}_\Delta(x)^{[p]} + (u-1)\overline{\rho}_\Delta(\delta(x))$

# Action on cohomology and its categorification

# $\mathbb{G}_m^\gamma$ acts on

- $\overline{\Delta}_X(Y)$
- $\overline{\Delta}_X(Y(1)) := \overline{\Delta}_X(Y \times_S Y)$
- $\cong \overline{\Delta}_X(Y) \times_Y \overline{\Delta}_Y(Y(1)) \ \cong \overline{\Delta}_Y(Y(1)) \times_Y \overline{\Delta}_X(Y)$
- $\overline{\Delta}_X(Y)(n)$ for all $n$

# $\mathbb{G}_m^{\gamma}$ acts on

- The prismatic Cech-Alexander complex:
- $\mathcal{O}_{\Delta_X(Y)} \to \mathcal{O}_{\Delta_X(Y(1))} \to \mathcal{O}_{\Delta_X(Y(2))} \to \mathcal{O}_{\Delta_X(Y(3))} \to \cdots$
- Hence on the prismatic cohomology of $X/k$.
- Can we find a DGA with an action?

# The prismatic Higgs complex

- DR complex: $d\colon \mathcal{O}_Y \to \Omega^1_{Y/S} \to \Omega^2_{Y/S} \to \cdots$
- $p$-DR complex: multiply all differentials by $p$.
- $d' := pd$ is a $p$-connection on $\mathcal{O}_Y$.
- Claim: $d'$ extends uniquely to a $p$-connection on $\mathcal{O}_{D_X(Y)}$ and on $\mathcal{O}_{\Delta_X(Y)}$.
- Thm: $(\mathcal{O}_{\Delta_X(Y)} \otimes \Omega^{\bullet}_{Y/S}, d')$ calculates the prismatic cohomology of $X/S$.

- Reduce mod $p$ to get:
- $(\mathcal{O}_{\overline{\Delta}_X(Y)} \otimes \Omega^\bullet_{Y/S}, \overline{d'})$ computes prismatic cohomology of $X/k$.
- Quandary: $\mathrm{G}_\mathrm{m}^\gamma$-action is ***not*** compatible with $\overline{d'}$.

# Action on categories:

- Prismatic crystals on $X/k$. (i.e. $p$-torsion prismatic crystals on $X/S$.)
- $O_{\overline{\Delta}_X(Y)}$-modules with (compatible) prismatic stratification:
  - $(E, \epsilon) : \epsilon : p_2^*(E) \to p_1^*(E)$
- $O_{\overline{\Delta}_X(Y)}$-modules with (compatible) q-nilpotent Higgs field:
  - $(E, \theta) : \theta : E \to E \otimes \Omega^1_{Y/S}$.
- $O_X$-modules with q-nilpotent integrable connection . (via the F-transform)

# $\mathbb{G}_m^\gamma$ acts on all these:

- e.g on the category of $\mathcal{O}_{\overline{\Delta}_X(Y)}$-modules with prismatic stratification:
- $p_1, p_2 \colon \overline{\Delta}_X(Y(1)) \to \overline{\Delta}_X(Y)$
- $(E, \epsilon) : \epsilon \colon p_2^*(E) \to p_1^*(E)$
- $r_\Delta(E, \epsilon) := (\tilde{E}, \tilde{\epsilon})$, where:
- $(\tilde{E}, \tilde{\epsilon}) := r_\Delta^*(\epsilon) \colon p_2^* r_\Delta^*(E) \to p_1^* r_\Delta^*(E)$

- Therefore, $\mathrm{G}_{\mathrm{m}}^{\gamma}$ acts on Higgs fields:
- $(E, \epsilon) \leftrightarrow (E, \theta)$
- $(\tilde{E}, \tilde{\epsilon}) \leftrightarrow (\tilde{E}, \tilde{\theta})$
- Describe: $(E, \theta) \mapsto (\tilde{E}, \tilde{\theta})$?
- Conjecture by Vologodsky, based on his construction of the action on $\mathrm{MIC}_{qn}(X/k)$

- If $\tilde{X} = Y$, $\tilde{E} = E$. What is $\tilde{\theta}$ ?
- $(E, \theta) \leftrightarrow E_\theta$, sheaf on cotangent space $\hat{\mathrm{T}}^*_{X/S}$.
- ``Obvious'' action $r_{\mathrm{T}}$ of $\mathbf{G}_{\mathrm{m}}$, also of $\mathbf{G}^\gamma_{\mathrm{m}}$, on $\hat{\mathrm{T}}^*_{X/S}$.
- Conjecture: $E_{\tilde{\theta}} = s_{\Delta *}(E_\theta)$, where
- $s_\Delta := \alpha^{-1} \circ r_{\mathrm{T}} \circ (\alpha \times \mathrm{id}_{\mathrm{G}^\gamma_{\mathrm{m}}})$.

# The $\alpha$-transform

- $\zeta\colon F^*_{\overline{Y}}(\Omega^1_{\overline{Y}/S}) \to \Omega^1_{\overline{Y}/S}$ induced by $p^{-1}\phi^*$.
- $\hat{\zeta}\colon T_{\overline{Y}/S} \to F^*_{\overline{Y}}(T_{\overline{Y}/S})$ (its dual)
- Write geometrically:
- $\mathrm{T}^{*(p)}_{\overline{Y}/S} \longrightarrow \mathrm{T}^*_{\overline{Y}/S}$
- $F_{\mathrm{T}^*/\overline{Y}} \uparrow \quad \nearrow z$
- $\mathrm{T}^*_{\overline{Y}/S}$

$$\alpha := \mathrm{id} - z$$

- $\alpha\colon T^*_{\overline{Y}/S} \to T^*_{\overline{Y}/S}$ is a group scheme morphism, restricts to $\mathbf{id}$ on the zero section
- $\hat{\alpha}\colon \hat{T}^*_{\overline{Y}/S} \to \hat{T}^*_{\overline{Y}/S}$ is an isomorphism,
- $\mathrm{HIG}_{qn}(\overline{Y}/S) \subset\ O_{\hat{T}^*_{\overline{Y}/S}}$-modules (full subcategory)
- $\alpha$ transform: $\mathrm{HIG}_{qn}(\overline{Y}/S) \to \mathrm{HIG}_{qn}(\overline{Y}/S) \quad (E,\theta) \mapsto \hat{\alpha}_*(E,\theta)$
- Equivalence of categories , inverse given by $\hat{\alpha}^*$
- Thm: $(E \otimes \Omega^{\bullet}_{\overline{Y}/S}, \theta^{\bullet}) \sim (\hat{\alpha}_*(E) \otimes \Omega^{\bullet}_{\overline{Y}/S}, \theta^{\bullet})$ (in derived category)

# Quandaries and resolutions

# Action on crystals

- Theorem: Vologodsky's conjecture is true.
    - Proof uses the formula for the $\mathbf{G}_{\mathrm{m}}^{\gamma}$- action on $\overline{\Delta}_Y(Y(1))$.
- It implies that the action on $\mathrm{MIC}_{\mathrm{qn}}(X/S)$ rescales the $p$-curvature,
- and hence agrees with his action,
- and that of Bhatt-Lurie (presumably).

# Why is $\overline{d'}$ not compatible with $r_\Delta$?

- $r_\Delta$ ***is*** compatible with the stratification of $\overline{\Delta}_X(Y)$.
- Review: relationship between stratifications and Higgs fields.
- and between stratifications and $p$-connections

- $Y \to Y(1)$ is an immersion of $\delta$-schemes.
- $p_1, p_2 \colon \Delta_Y(1) \to Y, \quad Y \to \Delta_Y(1)$
- $I_{Y/\Delta}$ is a PD-ideal.
- Have $I_{Y/\Delta}/I_{Y/\Delta}^{[2]} \cong \Omega^1_{Y/S}$ (involves mult. by $p$)

- $\mathbb{G}_{\mathrm{m}}^{\gamma}$ acts on $\overline{\Delta}_Y(1)$,
    - Action preserves $I_{\overline{Y}/\overline{\Delta}}$.
- But not $I^{[n]}_{\overline{Y}/\overline{\Delta}}$ for $n > 1$ and not even $I^{[2]}_{\overline{Y}/\overline{\Delta}}$.
    - Recall formula: $r_{\Delta}^{*}(\bar{\rho}_{\Delta}(x)^{[p]}) = \bar{\rho}_{\Delta}(x)^{[p]}) + (u-1)\bar{\rho}_{\Delta}(\delta(x))$.
- Hence not on $I_{\overline{Y}/\overline{\Delta}} I^{[2]}_{\overline{Y}/\overline{\Delta}} \cong \Omega^1_{\overline{Y}/S}$.

# Remedy

- Re: the $N^{\gamma}_{X/\overline{Y}}$ - action on $\overline{\Delta}_X(Y)$ gives us a connection:

  - $\nabla : \mathcal{O}_{\overline{\Delta}_X(Y)} \to \mathcal{O}_{\overline{\Delta}_X(Y)} \otimes \Omega^1_{\overline{D}/X}$

  - Compose with
    $\mathcal{O}_{\overline{\Delta}_X(Y)} \otimes \Omega^1_{\overline{D}/X} \cong \mathcal{O}_{\overline{\Delta}_X(Y)} \otimes I_{X/\overline{Y}}/I^2_{X/\overline{Y}} \to \mathcal{O}_{\overline{\Delta}_X(Y)} \otimes i_X^*(\Omega^1_{\overline{Y}/S})$

  - Get a new Higgs field: $d_\delta : \mathcal{O}_{\overline{\Delta}_X(Y)} \to \mathcal{O}_{\overline{\Delta}_X(Y)} \otimes \Omega^1_{\overline{Y}/S}$

- Theorem: $d_\delta$ is the $\alpha$-transform of $\overline{d}'$.
- Corollary: $(O_{\overline{\Delta}_X(Y)} \otimes \Omega^\bullet_{\overline{Y}/X}, d_\delta)$ also calculates the prismatic cohomology of $X/k$. (Already known to Bhatt).

- $d_\delta$ ***is*** compatible with the Sen operator, since $\theta_\Delta = \nabla_{\theta_D}$.

  - Integrability of $\nabla$ implies that the following diagram commutes:

    - $$\begin{array}{ccc} O_{\overline{\Delta}_X(Y)} & \xrightarrow{d_\delta} & O_{\overline{\Delta}_X(Y)} \otimes \Omega^1_{Y/S} \\ \theta_\Delta \downarrow & & \downarrow \theta_\Delta + 1 \\ O_{\overline{\Delta}_X(Y)} & \xrightarrow{d_\delta} & O_{\overline{\Delta}_X(Y)} \otimes \Omega^1_{Y/S} \end{array}$$

# Summary

# Envelopes are torsors:

$$\begin{array}{ccc}
\overline{\Delta}_X(Y) \times_X N^{\gamma}_{X/\overline{Y}} & \xrightarrow{\cong} & \overline{\Delta}_X(Y) \times_X \overline{\Delta}_X(Y) \\
\downarrow & & \downarrow \\
\overline{D}_X(Y) \times_X N_{X/\overline{Y}} & \xrightarrow{\cong} & \overline{D}_X(Y) \times_X \overline{D}_X(Y)
\end{array}$$

# A lifting of $X$ in $Y$ produces:

- A section of $\overline{D}_X(Y) \to X$
- An isomorphism $\overline{D}_X(Y) \cong N_{X/\overline{Y}}$
- An action $r_D$ of $\mathrm{G_m}$ on $\overline{D}_X(Y)$.
- An action $r_\Delta$ of $\mathrm{G}_\mathrm{m}^\gamma$ on $\overline{\Delta}_X(Y)$.

# Resolution of the quandary

- There are ***two*** natural Higgs fields on $\overline{\Delta}_X(Y)$, both of which calculate the prismatic cohomology of $X/\overline{S}$,
  - related by the $\alpha$-transform.
  - ***One*** of which is compatible with the $\mathrm{G}_{\mathrm{m}}^{\gamma}$-action induced by a lifting of $X \to Y$.
  - ***May*** be amenable to calculation.

# Thanks again, Bhargav!