A Prismatic Perspective on Hodge-de Rham Degeneration

May 16, 2023

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For such a surface, the Hodge-de Rham spectral sequence $\mathrm{H}^i(X, \Omega^j_X) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{dR}}(X)$ does not degenerate.

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In 1987, Deligne-Illusie found an algebraic proof, using characteristic p methods.

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The scheme U has many points s such that $\kappa(s)$ is a finite field!

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Assume that:

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Corollary

In characteristic zero, the Hodge-de Rham spectral sequence degenerates.

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Theorem (Petrov, 2023)

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Goal: sketch a "prismatic" proof of this result, based on joint work with Bhargav Bhatt (and discovered independently by Drinfeld).

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- $\Omega_X^{\not{D}}$ *has an endomorphism* Θ *which acts by* $-d$ *on* $\mathcal{H}^d(\Omega_X^{\not{D}})$.

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When X is projective, this implies the degeneration of the Hodge-de Rham spectral sequence (by dimension counting).

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With the splitting of the last slide, this (essentially) recovers the Hodge-Tate decomposition

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- We will replace X by its p-completion (a formal scheme over $\mathrm{Spf}(\mathbf{Z}_p)$).

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Extending scalars from A to A/I gives a coherent complex of $\mathcal{O}_Y$-modules $\overline{\mathbb{\Delta}}_{X/A}$.

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Specializing to $q = 1$ gives the usual de Rham complex $(\Omega^0_X \xrightarrow{d} \Omega^1_X \to \cdots)$.

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Warning

From this perspective, it is hard to see the Sen operator Θ!

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This carries a transitive action of the affine group scheme $W^{\times}$ (by multiplication). The action is not free: the equation

$$V(u) \cdot u' = V(u \cdot F(u')),$$

means that Y is stabilized by the subgroup

$$W^{\times}[F] := \ker(F : W^{\times} \to W^{\times}).$$

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This is the diffracted Hodge complex.

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That is, $W^\times[F] = \mathbf{G}_m^\sharp$ is the divided power envelope of $\mathbf{G}_m$ along its identity section.

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