The noncommutative minimal model program

Northwestern, May 19, 2023

Daniel Halpern-Leistner

Cornell University

Partially in collaboration with Alekos Robotis

Outline

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Structure of Derived Categories

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2. Unexpected symmetries, i.e., group actions on $D^b(X)$,
3. Unexpected decompositions of $D^b(X)$ into simpler pieces.

Example 1: D-equivalence conjecture

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MMP to reduce the problem to a single flop, good control over singularities, modular interpretation, ...

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Question

How common is this phenomenon?

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- Barlow surfaces (GGvBKS,'12).
- $\mathbb{P}^2$ blown up at 10 general points (Krah, '23).

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Provide a *mechanism* for many conjectures about $D^b(X)$ that is more direct than appealing to homological mirror symmetry.

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2. These paths are convergent in a partial compactification of $\mathrm{Stab}(X)/\mathbb{G}_a$ (In progress)
3. Noncommutative MMP $=$ conjectures about canonical paths on $\mathrm{Stab}(X)/\mathbb{G}_a$ that imply previous conjectures about $D^b(X)$.

Comparing definitions

Context: X smooth projective variety over $\mathbb{C}$. $\mathscr{C} = D^b(X)$.
Charge lattice $\Lambda := H^*_{\mathrm{alg}}(X) \subset H^*(X;\mathbb{C})$. Mukai vector map

$$v = (2\pi i)^{\deg/2}\mathrm{ch}\colon K_0(X) \twoheadrightarrow \Lambda.$$

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Stability condition:

Central charge homomorphism
$Z : \Lambda \to \mathbb{C}$ with

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Theorem (Bridgeland)

Forgetful map $\mathrm{Stab}(\mathscr{C}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$ taking $(\mathscr{P}_\bullet, Z) \mapsto Z$ is a local homeomorphism (for natural metric topology on Stab).

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$\Rightarrow$ Important observation: *Paths* in $\mathrm{Stab}(\mathscr{C})$ are determined by starting point and a path in $\mathrm{Hom}(\Lambda, \mathbb{C})$.

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Lemma (Key Lemma)

$\exists$ a SOD $\mathscr{C} = \langle \mathscr{C}_1, \ldots, \mathscr{C}_n \rangle$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, where $\Im(\alpha_1) < \ldots < \Im(\alpha_n)$ and

$$\mathscr{C}_i \subset \mathscr{C} \text{ is generated by eventually semistable } E \text{ with } \alpha_E = \alpha_i.$$

Furthermore each $\mathscr{C}_i$ **admits a stability condition** whose semistable objects are eventually semistable and $Z_i(E) = e^{\beta_E}$.

Key lemma

Proof idea.

Let $G_j := \mathrm{gr}_j E$ for the eventual HN filtration of E. Then $\phi_t(G_j) \sim \Im(\alpha_{G_j} t + \beta_{G_j})/\pi$ is increasing in j for all $t \gg 0$, so $\Im(\alpha_{G_j})$ is increasing in j. The filtration for the SOD is the coarsening of this filtration that groups terms with the same α. □

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Proposition (Partial converse to key lemma)

Any SOD where all the factors admit stability conditions can be recovered from a quasi-convergent path. (Because $\mathscr{C}$ is smooth and proper)

↳ Uses Collins-Polishchuk gluing construction

A proposal

Folklore categorical analogy

(stability condition on $D^b(X)$) $\leftrightarrow$ (ample divisor class on X)

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Principle

Categorical birational geometry = the study of SOD's of $D^b(X)$ in which every factor admits a stability condition.

$\uparrow$

"polarizable" SOD's

Example: no phantoms

Lemma

If $\mathscr{C}$ is smooth and proper, $\dim(K_0(\mathscr{C}) \otimes \mathbb{Q}) = 1$, and $\mathscr{C}$ admits a stability condition, then $\mathscr{C}$ is generated by a single exceptional object.

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So, if SOD is “polarizable” and it looks like it comes from a full exceptional collection on the level of K-theory, then it does.

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Example

On the Barlow surface, $D^b(X) = \langle L_1, \ldots, L_{10}, {}^{\perp}\{L_1, \ldots, L_{10}\}\rangle$ can not arise from a quasi-convergent path in $\mathrm{Stab}(X)$.

Plan for the remainder of the talk

1. "Bordification" of $\mathrm{Stab}(\mathscr{C})/\mathbb{G}_a$
2. Formulate the noncommutative minimal model program
3. Discuss consequences

What is going on in key lemma?

Fix E and consider the configuration $\{\log Z_t(\mathrm{gr}_i^{HN}(E))\}_{i=1}^n$ in $\mathbb{C}$:

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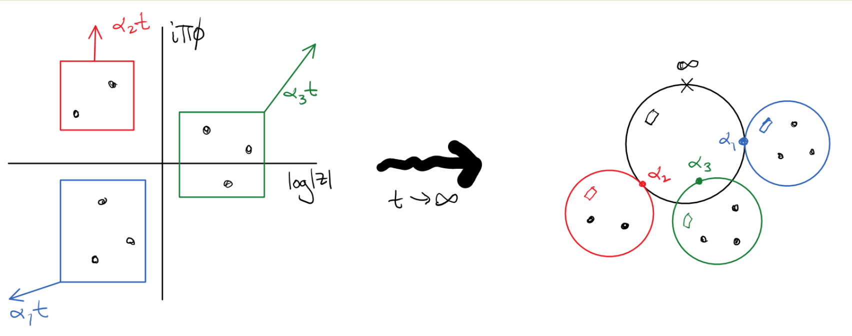

$(\mathbb{P}^1, dz)$ degenerates to a *multi-scaled line*: a marked genus 0 nodal curve with meromorphic differential (Σ, Ω) with all components isomorphic to $(\mathbb{P}^1, dz)$. (also has a "level structure")

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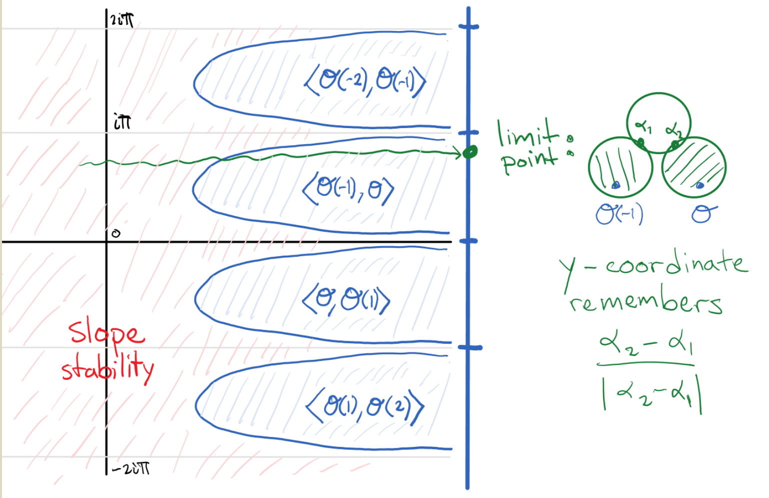
Regard $\log$ of central charge of σ_i as taking values in the corresponding terminal component of Σ.

(Equivalence relation on generalized stability conditions is slightly non-trivial.)

Example of $\mathbb{P}^1$

$\mathrm{Stab}(\mathbb{P}^1)/\mathbb{G}_a \cong \mathbb{C}$. Partially compactified by the blue vertical line at infinity. Green path is quasi-convergent.



The space of generalized stability conditions

In progress (joint with Alekos Robotis):

- Constructing a Hausdorf space $\overline{\mathbb{P}\mathrm{Stab}}(\mathscr{C})$ containing $\mathrm{Stab}(\mathscr{C})/\mathbb{G}_a$ as a dense open subset.

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- $\exists$ s.n.c. compactification $\mathbb{C}^n/\mathbb{G}_a \subset M_n^{ms}$ by n-marked stable multi-scaled lines. $\tilde{M}_n^{ms} :=$ real oriented blowup of M_n^{ms}

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- $\exists$ locally defined continuous maps

$$\log Z : U \subset \overline{\mathbb{P}\mathrm{Stab}}(\mathscr{C}) \to \tilde{M}_n^{ms}$$

- **Conjecture:** the $\log Z$ maps are local homeomorphisms, making $\overline{\mathbb{P}\mathrm{Stab}}(\mathscr{C})$ a manifold with corners.

The NMMP conjectures (arXiv:2301.13168)

Simplified, absolute version:

A. To any smooth projective X, one can associate a canonical collection of quasi-convergent paths $\sigma_t^\psi \in \mathrm{Stab}(X)/\mathbb{G}_a$, and different generic parameters ψ give mutation equivalent SOD's

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B. If $\pi : X \to X'$ is a birational morphism of smooth projective varieties, then for suitable parameters, the SOD for X refines the SOD obtained by combining

$$D^b(X) = \langle \ker(\pi_*), \pi^*(D^b(X')) \rangle$$

with the SOD of $D^b(X') \cong \pi^*(D^b(X'))$.

Consequences

Assuming the NMMP conjectures:

Proposition

Given a smooth projective X with $h^0(K_X) > 0$, $\exists$ an admissible category $\mathscr{M}_X \subset D^b(X)$, supported on all of X, such that for any X' that is birational to X, one has an admissible embedding $\mathscr{M}_X \subset D^b(X')$.

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$\mathscr{M}_X$, the *noncommutative minimal model*, is a birational invariant of X.

Corollary

If $X \dashrightarrow X'$ and $|K_X|$ is baspoint free, then $\exists$ admissible embedding $D^b(X) \hookrightarrow D^b(X')$, which is an equivalence if $|K_{X'}|$ is also basepoint free.

Illustrating the idea in an example.

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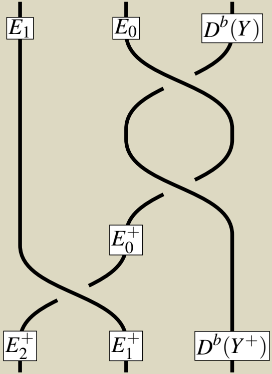
Mutation functor gives an equivalence $D^b(Y) \cong D^b(Y^+)$.



More precise proposal for canonical paths

QDE Proposal: $\exists$ quasi-convergent paths in $\mathrm{Stab}(X)/\mathbb{G}_a$ with central charge

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with $\Phi_t \in \mathrm{End}(H^*_{\mathrm{alg}}(X)_{\mathbb{C}})$ a fundamental solution of a (truncated) quantum differential equation for $\xi(t) \in H^*_{\mathrm{alg}}(X)_{\mathbb{C}}$

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$$0 = t\frac{d\xi}{dt} + z^{-1} c_1(X) \star_{\psi+\ln(t)} \xi. \qquad (1)$$

Spanning Condition

(Informal version) Any asymptotic class of solutions of (1) is spanned by $\Phi_t(v_E)$ for some eventually semistable E.

Relationship to Dubrovin / Gamma conjectures

Proposition

$D^b(X)$ admits a full exceptional collection if:

- $Ch : K_0(X) \otimes \mathbb{C} \to H^*(X;\mathbb{C})$ is bijective;
- The QDE Proposal and Spanning Condition hold; and
- The eigenvalues of $c_1(X) \star_\psi (-)$ are distinct.

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Example (It works for $D^b(\mathbb{P}^1)$)

Iritani's "quantum cohomology central charge" $Z_{t,\psi}(E)$ lifts to a path in $\mathrm{Stab}(\mathbb{P}^1)/\mathbb{G}_a \cong \mathbb{C} \cong H^2(\mathbb{P}^1;\mathbb{C})$ that starts at ψ and moves straight to the right.

Relationship to blowup formula

One can recover the Hodge structure on $K^{\mathrm{top}}(X)$ from $D^b(X)$.

Decategorification

Any SOD of $D^b(X) \rightsquigarrow$ Direct sum decomposition of the Hodge structure on $K_0^{top}(X)$

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Question (Hodge theoretic MMP)

Can one see these decompositions directly from truncated QDE?

↰ Alternative version of the Katzarkov-Kontsevich-Pantev-Yu blowup formula conjecture.