Ko Aoki.

FRG workshop.

18 March 2026.

(Commutative) higher motives.

number theory

arithmetic geometry

systems of poly eqs

$\wr$

varieties$/\mathbb{Z}$.

cohomology theories

field

Betti	$H^*(X(\mathbb{C}); e)$
étale	$H^*(X; \mathbb{F}_\ell)$
	$H^*_{\acute{e}t}(X; \mathbb{Q}_\ell)$
de Rham	$H^*_{dR}(X/k)$
rigid	
$\vdots$	

Comparison theorems
but different info.

Idea (Grothendieck).

Motives.

Ex. Morel–Voevodsky.

Coefficient theories.

$Sh_{Betti}(X(\mathbb{C}))$ 6 operations

DMod

Idea (Scholze)

$\{\text{varieties}/k\} \longrightarrow 2\mathrm{Mot}(k)$.

$\langle \{\text{varieties}/\mathbb{Z}\} \rangle$

$2SH'(\mathbb{Z})$

sm. base chage
excision
$\mathbb{A}^1$-invariance
Tate invertibility.

finely gen as stable presentable 2-cat. ($\$$-linear)

Category of kernels.
Drinfeld
Liu-Zheng
Fargues-Scholze.

Shv: six ops $\rightsquigarrow$ $2Shv(*)$ is generated by

$\mathrm{Cor}(\mathcal{G}) \longrightarrow \mathrm{Pr}$. $\qquad [X] \, (= |Shv(X|*)) \text{ for } X \in \mathcal{G}$

Thm (Ayoub, Drew-Gallauer, Scholze, Aoki). $2SH'(\mathbb{Z}) \simeq 2SH(\mathbb{Z})$.

Ring stacks.

Ex (Simpson). $X/\mathbb{Q}$ variety $\qquad DMod(X) \simeq QCoh(X^{dR})$.

Drinfeld. To get $X \longrightarrow X^{dR}$ just need $(\mathbb{A}^1)^{dR}$ as ring stack.

Coefficient theory $\longleftrightarrow$ ring stacks

2-motives

Thm (A.). $2SH(\mathbb{Z})$ is freely generated by homologically trivial smooth separated (1-affine) ring stack.

$[*] \longrightarrow [\mathbb{A}^1]$ fully faithful — excision for $[\mathbb{G}_m]$ $[\mathbb{A}^1]$ $[0]$ — sm. base change for $\mathbb{A}^1$. — Tate stability

$2SH(\mathbb{Z}) \longrightarrow \mathcal{C}$ stable pres. sym. mon. 2-cat

$[\mathbb{A}^1] \longmapsto R \in \mathrm{Ring}(\mathrm{CAlg}(\mathcal{C})^{op})$.

Thm (A.). $2SH_{et}(\mathbb{Z})$ is freely generated by ——— ring stack (étale sheafified)

Kummer $R[1/\ell] \xrightarrow{x \mapsto x^\ell} R[1/\ell]$ cov.

Artin-Schreier $R/p \xrightarrow{x \mapsto x^p - x} R/p$ cov.

Analytic geometry.

Berkovich geometry: building blocks are Banach rings.

Scholze: Berkovich motives $D_{mot}(-; \mathbb{S})$.

Categorically nice, hard to get realizations.

Thm (A.). $2D_{mot}(\mathbb{Z})$ is freely generated by KAS (Kummer, Artin-Schreier) hom. triv. sm. ring stack with abs. value $R \longrightarrow [0,\infty)$ with hom. triv. open unit disk.

$\mathcal{O}(R) \longleftarrow \mathrm{Shv}([0,\infty))$

Applications.

- Fundamental theorem of algebra.

- New realizations like analytic de Rham $/\mathbb{Q}_p$ as in Rodríguez Camargo.

- Analytic Habiro cohomology (Scholze).

- 2-rigidity.

<u>Thm</u> (A.). $2D_{mot}(\mathcal{O})$ totally imaginary number ring is 2-rigid. $\Longleftrightarrow$ 0-proper (like compact Hausdorff space)

Scholze – Stefanich Gestalten.

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{M} = \mathrm{Spec}(2D_{mot}(\mathbb{Z})). \\ \downarrow & & \\ \text{1-affine ring objects in } X. & & \end{array}$$

Gives map from compactification.
Algebraically gives automatic nuclearity.