A Universal Property of the Gray Tensor - Townsend

(March 16, Ea E T)

Introduction

This note follows a talk by Townsend on a *universal property of the gray tensor*[1] at the FRG workshop on higher categories and geometry.

Motivational Principle

Slogan: "Things get simpler at ∞"

- **E.g.** Higher affineness

Analogy: "Things get simpler at $(\infty, 1)$"

- **E.g.** An $(\infty, 1)$-topos X is a locally presentable $(\infty, 1)$-category such that $X_{/(-)} : X^{op} \to (\infty, 1)\mathrm{Cat}$ preserves limits.

Problem

1. Things seem complicated!
2. Things don't seem simpler
   1. **E.g.** The description $(\infty, \infty)\mathrm{Cat} = \lim_n (\infty, n)\mathrm{Cat} = \mathrm{colim}_n^{\mathrm{Pr}^L} (\infty, n)\mathrm{Cat}$ doesn't make the world of (∞, ∞)-categories simpler than just a pile of (∞, n)-categories, for $n \geq 0$.

Solution?

How do (∞, m)-categories and (∞, n)-categories interact?

Examples of Relations between $(\infty, m)\mathrm{Cat}$ and $(\infty, n)\mathrm{Cat}$

- **(1)** We have the inclusion $(\infty, n)\mathrm{Cat} \hookrightarrow (\infty, n+1)\mathrm{Cat}$
- **(2)** We have the suspension $\Sigma : (\infty, n)\mathrm{Cat} \to (\infty, n+1)\mathrm{Cat}$
- **(3)** We have the Gray tensor product $(\infty, m)\mathrm{Cat} \times (\infty, n)\mathrm{Cat} \xrightarrow{\boxtimes} (\infty, m+n)\mathrm{Cat}$
- **(4)** We have the join $(\infty, m)\mathrm{Cat} \times (\infty, n)\mathrm{Cat} \xrightarrow{\star} (\infty, m+1+n)\mathrm{Cat}$

Key Idea: We want to organize the data of (∞, ∞)-categories using the *gray tensor product* and *join*.

Different Gray Tensors

- **Complicial (Verity):** The gray tensor product of complicial sets A and B, denoted $A \boxtimes B$, has underlying simplicial set $A \times B$, with a complicated combinatorial description of the markings.
- **Comical (Campion, Doterty, Kapulkin, Maehara[2]):** Certain marked cubical sets, where for A, B cubical sets, $A \otimes B$ as an underlying cubical set is given naturally in terms of the cubical structure, and a cell $a \otimes b$ is marked if a or b is.

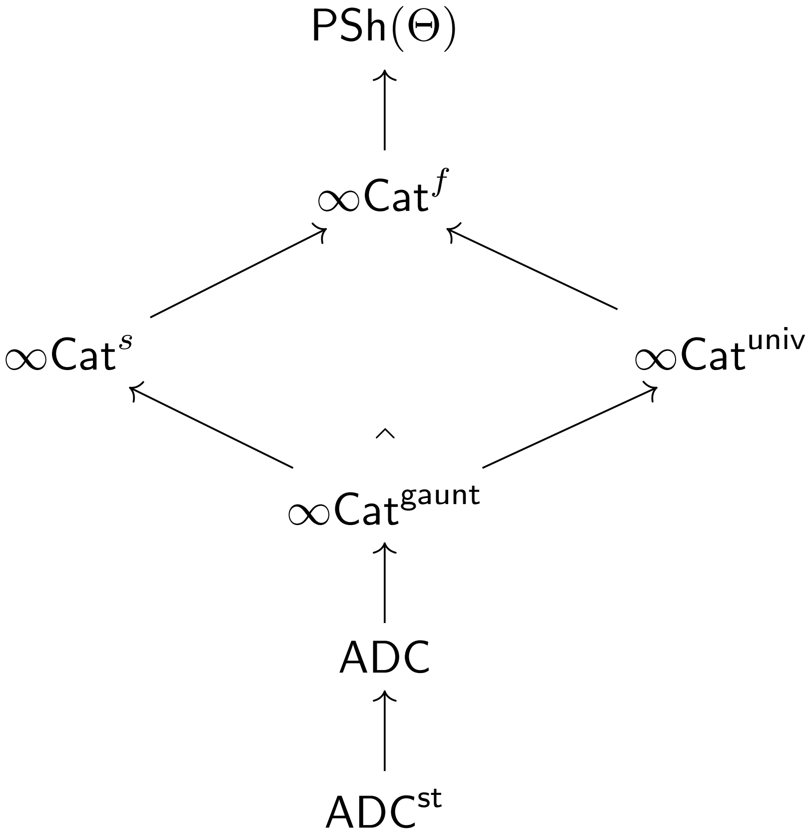
Approach (Maehara, Campion)

Import the Gray tensor product $\boxtimes$ from (strict) ∞-categories, with description in terms of augmented directed complexes due to Steiner (c.f. Strict Inf-Cats). Specifically, one can show that $(\infty, \infty)\mathsf{Cat}$ is a localization of space-valued presheaves on (the (strict) 1-category of) augmented directed complexes (really the *strong Steiner complexes*), and then import the gray tensor product from the natural description of the gray tensor product on augmented directed complexes.

Gaunt (∞, ∞)-category

An (∞, ∞)-category $\mathcal{C}$ is **gaunt** if the space $\mathrm{Map}(\mathrm{D}^n, \mathcal{C})$ is 0-truncated for all n.

Intuition: Gaunt (∞, ∞)-categories are those with no non-trivial equivalences. In particular, all gaunt (∞, ∞)-categories are **strict** ∞-categories. We have a pullback diagram



so that the *Gaunt* (∞,∞)-categories are precisely the *strict univalent* (∞,∞)-categories. Here $(\infty,\infty)\mathsf{Cat} =: \infty\mathsf{Cat}^{\mathsf{univ}}$ are the univalent (∞,∞)-categories. The Strong Steiner complexes admit a natural description of the Gray tensor product, which through Day convolution induces a monoidal structure on pre-sheaves on Strong Steiner complex.

- **Key Point:** This result is an instance of the **Day reflection theorem**.

> **Question**
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> What needs to be shown for this to descend along the localizations?

- **Answer:** If L is a localization functor from a monoidal $(\infty,1)$-category, we need that if $f : A \to B$ is a map such that Lf is an equivalence, then for any object C, $L(C \otimes f)$ and $L(f \otimes C)$ are also equivalences.

> **Question**
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> But what is ⊠ *realllllllllly*?

> **Conjecture**
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> Let $(\mathcal{A}, \otimes, \mathbb{1}) \in \mathsf{Alg}_{\mathbb{E}_1}(\mathsf{Pr})$. Then we have a pullback diagram

$$\begin{array}{ccc} \mathsf{Fun}^{L,\mathbb{E}_1}\Big(((\infty,\infty)\mathsf{Cat},\boxtimes),(\mathcal{A},\otimes)\Big) & \longrightarrow & \mathsf{Fun}^{L,\mathbb{E}_0}\Big((\infty,2)\mathsf{Cat},\mathcal{A}\Big) \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{Fun}^{L,\mathbb{E}_0}\Big((\infty,1)\mathsf{Cat},\mathcal{A}\Big) & \longrightarrow & \mathsf{Fun}^{L,\mathbb{E}_0}\Big((\infty,1)\mathsf{Cat}\otimes(\infty,1)\mathsf{Cat},\mathcal{A}\Big) \end{array}$$

where $(\infty,1)\mathsf{Cat}\otimes(\infty,1)\mathsf{Cat}\simeq\infty\mathsf{DblCat}$, and $\infty\mathsf{DblCat}\to(\infty,2)\mathsf{Cat}$ is *left adjoint to the squares functor*. Here the bottom map sends $\Gamma:(\infty,1)\mathsf{Cat}\to\mathcal{A}$ to the composite

$$(\infty,1)\mathsf{Cat}\otimes(\infty,1)\mathsf{Cat}\xrightarrow{\Gamma\otimes\Gamma}\mathcal{A}\otimes\mathcal{A}\xrightarrow{\otimes}\mathcal{A}$$

using the monoidal structure on $\mathcal{A}$.

- **Note:** A functor being $\mathbb{E}_1$ means it is *strict monoidal*, while being $\mathbb{E}_0$ means it preserves the unit.

Intuition: we need only specify the data of a cocontinuous functor $(\infty,1)\mathsf{Cat}\to\mathcal{A}$ functor such that the composite

$$\infty\mathsf{DblCat}\simeq(\infty,1)\mathsf{Cat}\otimes(\infty,1)\mathsf{Cat}\xrightarrow{\Gamma\otimes\Gamma}\mathcal{A}\otimes\mathcal{A}\xrightarrow{\otimes}\mathcal{A}$$

factors through the left adjoint $\infty\mathsf{DblCat}\to(\infty,2)\mathsf{Cat}$ to the squares functor $\mathsf{Sq}:(\infty,2)\mathsf{Cat}\to\infty\mathsf{DblCat}$.

- **Key Point:** The left adjoint to Sq is the restriction of the Gray tensor product $\boxtimes:(\infty,1)\mathsf{Cat}\otimes(\infty,1)\mathsf{Cat}\to(\infty,2)\mathsf{Cat}$.

Observations on the Conjecture

- An oriented category (in the sense of Gepner-Heine) is a module over $((\infty,\infty)\mathsf{Cat},\boxtimes)$, which can be realized as a left adjoint functor

  $$((\infty,\infty)\mathsf{Cat},\boxtimes)\to\mathsf{End}(\mathcal{M})$$

  The conjecture says that this consists of the data of
- **(i)** a left adjoint functor $\Gamma:(\infty,1)\mathsf{Cat}\to\mathsf{End}(\mathcal{M})$, which is determined by the actions on Δ, namely by a co-Segal object $\Gamma|_\Delta:\Delta\to\mathsf{End}(\mathcal{M})$
- **(ii)** a left adjoint functor $\Gamma\otimes\Gamma:\infty\mathsf{DblCat}\to\mathsf{End}(\mathcal{M})$, determined by a bi-cosimplicial Segal object $\Gamma\otimes\Gamma|_{\Delta\times\Delta}:\Delta\times\Delta\to\mathsf{End}(\mathcal{M})$ which is dual to an object arising from the squares functor construction.

Recall: The squares functor is induced by a functor $\Delta \times \Delta \to \Theta_2$ sending $([n], [m])$ to the Gray tensor product $[n] \boxtimes [m]$.

> **Question**
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> What is the analogous universal property for the gray join?

References

1. Campion, Timothy. “The Gray Tensor Product of (∞, n)-Categories.” arXiv:2311.00205. Preprint, arXiv, November 1, 2023. https://doi.org/10.48550/arXiv.2311.00205 . ↩
2. Campion, Tim, Chris Kapulkin, and Yuki Maehara. “A Cubical Model for (∞, n)-Categories.” *Geometry & Topology* 29, no. 3 (2025): 1115–70. https://doi.org/10.2140/gt.2025.29.1115 . ↩