Higher Algebraic Geometry - Stefanich

(March 16-18, Ea E T)

Introduction

These notes follow a sequence of talks by Stefanich on his work with Scholze on *Higher Algebraic Geometry*[1] [2] [3] at the FRG workshop on higher categories and geometry.

Plan

1. Categorical background
2. Category of "*Gestalten*"
3. Applications

Setup: Let R be a (commutative) ring. Then Mod_R will denote the symmetric monoidal stable $(\infty, 1)$-category of R-modules.

- **Key Idea:** We think of Mod_R as a categorical object generalizing the underlying ring R. Note that if $\mathbb{1}$ denotes the monoidal unit, $\mathsf{End}_{\mathsf{Mod}_R}(\mathbb{1}) = R$ recovers the underlying ring. If S is another ring, then we have an identification

$$\mathsf{Hom}(R, S) \simeq \mathsf{Fun}^{L,\otimes}(\mathsf{Mod}_R, \mathsf{Mod}_S)$$

between the set of maps of rings and the space symmetric monoidal colimit preserving functors between their module $(\infty, 1)$-categories. Given a map of rings $(R \to S)$, extension by scalars gives a functor

$$- \otimes_R S : \mathsf{Mod}_R \to \mathsf{Mod}_S$$

Further, to go backwards we can consider the induced map

$$R \simeq \mathsf{End}_{\mathsf{Mod}_R}(\mathbb{1}) \xrightarrow{-\otimes_R S} \mathsf{End}_{\mathsf{Mod}_S}(\mathbb{1}) \simeq S$$

Fully-Faithfulness of Embedding Rings into Cocomplete Symmetric Monoidal $(\infty, 1)$-categories

The above remarks are summarized by the statement that we have a fully-faithful embedding:

$$\mathsf{Mod}_{(-)} : \mathsf{CRing} \hookrightarrow \mathsf{CoCompSymMon}(\infty, 1)\mathsf{Cat}$$

Example

If k is a ring, then

$$\mathsf{QCoh}(B\mathbb{G}_{m,k}) = \mathsf{Rep}(\mathbb{G}_{m,k}) = \mathsf{Mod}_k^{\mathbb{Z}}$$

is the $(\infty,1)$-category of $\mathbb{Z}$-graded k-modules. Since the unit *does not* generate this $(\infty,1)$-category under colimits, it is *not* in the image of the $\mathsf{Mod}_{(-)}$ embedding.

We have a similar embedding

$$\mathsf{CoCompSymMon}(\infty,1)\mathsf{Cat} \hookrightarrow \mathsf{CoCompSymMon}(\infty,2)\mathsf{Cat}$$

where on the right *cocomplete* refers to $(\infty,2)$-categories which have cocomplete underlying $(\infty,1)$-categories with cocomplete $(\infty,1)$-categorical morphism categories. That is, the objects are in particular cocomplete $(\infty,1)$-categories enriched in cocomplete $(\infty,1)$-categories (**informal**).

- **Intuition:** We should have both pseudo and (op)lax colimits.

Colimits in Enriched Higher Category Theory

Throughout let $\mathcal{C}$ be a symmetric monoidal $(\infty,1)$-category.

⚠ Warning

When considering cocompletions, and cocomplete higher categories, we have to be careful and *explicit* about *size issues*. In this talk this is done by fixing some uncountable regular cardinal κ, so that cocompleteness will refer to κ-cocompleteness, which is to say the existence of colimits for κ-small diagrams.

Due to these considerations, we more precisely assume $\mathcal{C} \in \mathsf{CAlg}(\mathsf{Pr}_\kappa^L)$, which is to say $\mathcal{C}$ is a symmetric monoidal locally κ-presentable $(\infty,1)$-category. In particular, this means that

1. $\mathcal{C}$ has colimits (and limits)
2. $\mathcal{C}$ is generated under colimits by a small collection of κ-compact objects
3. The $\otimes$ is compatible with colimits and κ-compactness
   For example, $\mathcal{C}$ could be Mod_k for any ring k.

Nota: We write $\mathcal{C}^\kappa \subseteq \mathcal{C}$ for the full subcategory on κ-compact objects. Then the above hypotheses ensure that the symmetric monoidal structure on $\mathcal{C}$ restricts to a symmetric monoidal structure on $\mathcal{C}^\kappa$.

κ-cocomplete $\mathcal{C}$-enriched Category

A **κ-cocomplete $\mathcal{C}$-enriched category** is an $(\infty,1)$-category $\mathcal{D}$ with κ-small colimits equipped with an action

$$\mathcal{C}^\kappa \times \mathcal{D} \to \mathcal{D}$$

which preserves κ-small colimits in each variable.

Enrichment from Action

For any $x, y \in \mathcal{D}$, we have $\underline{\mathrm{Hom}}_{\mathcal{D}}(x, y) \in \mathcal{C}$ such that for any $c \in \mathcal{C}^\kappa$

$$\mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}_{\mathcal{D}}(x, y)) \simeq \mathrm{Hom}_{\mathcal{D}}(c \otimes x, y)$$

Nota: We write Rex_κ for the $(\infty, 1)$-category of small $(\infty, 1)$-categories with κ-small colimits. This $(\infty, 1)$-category has the *Lurie tensor product* making it symmetric monoidal, where this monoidal structure can be expressed by the property that for any $\mathcal{D}, \mathcal{E} \in \mathrm{Rex}_\kappa$, we have a *universal* functor

$$\otimes : \mathcal{D} \times \mathcal{E} \to \mathcal{D} \otimes \mathcal{E}$$

which is the **initial functor which is κ-cocontinuous in each variable**. As a consequence

$$\mathrm{Rex}_\kappa(\mathcal{C}) := \mathrm{Mod}_{\mathcal{C}^\kappa}(\mathrm{Rex}_\kappa)$$

has objects given precisely by the κ-cocomplete $\mathcal{C}$-enriched categories.

Generalizing the $\mathrm{Mod}_{(-)}$ Construction to Enrichment

For $A \in \mathrm{CAlg}(\mathcal{C})$, we have a natural action

$$\mathcal{C}^\kappa \times \mathrm{Mod}_A(\mathcal{C})^\kappa \to \mathrm{Mod}_A(\mathcal{C})^\kappa, \quad (M, N) \mapsto M \otimes N$$

so that $\mathrm{Mod}_A(\mathcal{C})^\kappa \in \mathrm{Rex}_\kappa(\mathcal{C})$, and in fact this $\mathrm{Mod}_A(\mathcal{C})^\kappa \in \mathrm{CAlg}(\mathrm{Rex}_\kappa(\mathcal{C}))$. From this construction we get an *embedding*

$$\mathrm{CAlg}(\mathcal{C}) \hookrightarrow \mathrm{CAlg}(\mathrm{Rex}_\kappa(\mathcal{C}))$$

Key Idea: For $A \in \mathrm{CAlg}(\mathcal{C})$, we can form BA, which is a $\mathcal{C}$-enriched category with one object and $\mathrm{End}(\bullet) = A$. Adding κ-small colimits we obtain $\mathrm{Mod}_A(\mathcal{C})^\kappa$. In this way we obtain an adjunction

$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{C}) \arrow[rr, hook, bend right] \arrow[rr, phantom, "\top"] & & \mathrm{CAlg}(\mathrm{Rex}_\kappa(\mathcal{C})) \arrow[ll, bend right] \end{tikzcd}$$

where the right adjoint sends $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Rex}_\kappa(\mathcal{C}))$ to the endomorphism object of the unit $\underline{\mathrm{End}}_{\mathcal{D}}(\mathbb{1})$.

Local κ-presentability of $\mathrm{Rex}_\kappa(\mathcal{C})$

> The $(\infty, 1)$-category $\mathsf{Rex}_\kappa(\mathcal{C})$ is locally κ-presentable and symmetric monoidal.

Proof Idea.

We use a number of *free-forgetful* adjunctions to reduce to local κ-presentability of a more basic object. First, we have an adjunction given by forgetting or freely adding enrichment:

$$\begin{tikzcd} \mathsf{Rex}_\kappa \arrow[rr, hook, bend right] \arrow[rr, phantom, "\top"] & & \mathsf{Rex}_\kappa(\mathcal{C}) \arrow[ll, bend right] \end{tikzcd}$$

One also uses the free κ-cocompletion adjunction

$$\begin{tikzcd} (\infty, 1)\mathsf{Cat} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \mathsf{Rex}_\kappa \arrow[ll, hook, bend left] \end{tikzcd}$$

> **Important**
>
> This result allows us to iterate!!!

Nota: We write $1\mathsf{Rex}_\kappa := \mathsf{Rex}_\kappa$, and inductively, we define κ-cocomplete $(\infty, n+1)$-categories as objects in the $(\infty, 1)$-category

$$(n+1)\mathsf{Rex}_\kappa := \mathsf{Rex}_\kappa(n\mathsf{Rex}_\kappa)$$

> **Size-Considerations Shift**
>
> Moving forward we use a hierarchy of Grothendieck universes, which at the first three levels correspond to *small*, *large*, and *very large* spaces. κ-small will now correspond to small, while Rex_κ will be the *very large* $(\infty, 1)$-category of *large* $(\infty, 1)$-categories with *small* colimits. The κ-compact objects in Rex_κ correspond to the large locally presentable $(\infty, 1)$-categories.
>
> - **Note:** $\mathsf{Rex}_\kappa^\kappa$ is equivalent to a *large* $(\infty, 1)$-category, even though Rex_κ is *very large*

Nota: Moving forward we write $\mathsf{Pr} := \mathsf{Rex}_\kappa^\kappa$ and $\widehat{\mathsf{Pr}} := \mathsf{Rex}_\kappa$, and similarly $n\mathsf{Rex}_\kappa =: n\widehat{\mathsf{Pr}}$ consists of large (∞, n)-categories with small colimits, while the κ-compact objects $n\mathsf{Rex}_\kappa^\kappa =: n\mathsf{Pr}$ consists of locally presentable (∞, n)-categories.

Generalizing the $\mathsf{Mod}_{(-)}$ story

We have a sequence of adjunctions with fully-faithful left adjoints:

$$\begin{tikzcd}
\mathsf{CAlg}(\widehat{\mathsf{Pr}}) \arrow[r, hook, bend left, "\mathsf{Mod}_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{CAlg}(2\widehat{\mathsf{Pr}}) \arrow[l, bend left, "\underline{\mathsf{End}}_{(-)}(1)"] \arrow[r, hook, bend left, "\mathsf{Mod}_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{CAlg}(3\widehat{\mathsf{Pr}}) \arrow[l, bend left, "\underline{\mathsf{End}}_{(-)}(1)"] \arrow[r, hook, bend left, "\mathsf{Mod}_{(-)}"] \arrow[r, phantom, "\perp"] & \cdots \arrow[l, bend left, "\underline{\mathsf{End}}_{(-)}(1)"]
\end{tikzcd}$$

Categorical Ring

A **categorical ring** is a sequence $R = (R_1, R_2, R_3, \dots)$, with $R_n \in \mathsf{CAlg}(n\widehat{\mathsf{Pr}})$ such that

$$R_n := \underline{\mathsf{End}}_{R_{n+1}}(\mathbb{1})$$

That is, the $(\infty, 1)$-category of categorical rings is the **limit**

$$(\infty,\infty)\mathsf{CatRing} := \lim \Big(\mathsf{CAlg}(\widehat{\mathsf{Pr}}) \xleftarrow{\underline{\mathsf{End}}_{(-)}(\mathbb{1})} \mathsf{CAlg}(2\widehat{\mathsf{Pr}}) \xleftarrow{\underline{\mathsf{End}}_{(-)}(\mathbb{1})} \mathsf{CAlg}(3\widehat{\mathsf{Pr}}) \xleftarrow{\underline{\mathsf{End}}_{(-)}(\mathbb{1})} \cdots$$

Example

We have embeddings

$$\mathsf{CRing} \hookrightarrow \mathsf{CAlg}(\widehat{\mathsf{Pr}}) \hookrightarrow (\infty,\infty)\mathsf{CatRing}$$

Being algebraic geometry minded, we think of $\mathsf{CRing} = \mathsf{AffSch}^{op}$, and can consider the yoneda embedding $\mathsf{AffSch}^{op} \hookrightarrow \mathsf{PSh}(\mathsf{AffSch})^{op}$. Taking a right Kan extension along this embedding gives a functor

$$\begin{tikzcd}
\mathsf{AffSch}^{op} = \mathsf{CRing} \arrow[r, hook, "\mathsf{Mod}_{(-)}"] \arrow[dr, hook] & \mathsf{CAlg}(\widehat{\mathsf{Pr}}) \arrow[r, hook] & (\infty,\infty)\mathsf{CatRing} \\
& \mathsf{PSh}(\mathsf{AffSch})^{op} \arrow[ur, "\mathsf{Ran}"'] \arrow[u, phantom, "\Leftarrow"] &
\end{tikzcd}$$

where we write $\underline{\mathsf{QCoh}}(-)$ for the right Kan extension. Explicitly, for a pre-sheaf X on affine schemes,

$$\underline{\mathsf{QCoh}}(X) = (\mathsf{QCoh}(X), 2\mathsf{QCoh}(X), 3\mathsf{QCoh}(X), \dots)$$

If X is *affine*, $2\mathsf{QCoh}(X) = \mathsf{Mod}_{\mathsf{QCoh}(X)}$. In general, the data of the sequence is *not* determined at a finite stage.

- **Note:** $\underline{\mathsf{QCoh}}(-)$ is **not** fully-faithful in general (it is on qcqs schemes)

Note: If we use $(\infty, n)\mathsf{Cat}$ instead of $n\widehat{\mathsf{Pr}}$, we obtain **categorical spectra** instead of categorical rings.

Gestalten

We can now begin considering *Gestalten*.

Gestalten

The category of **Gestalten** is

$$\mathsf{Gest} := \mathsf{CatRing}^{op}$$

Note: We have an embedding $\mathbb{E}_\infty\mathsf{Ring} \hookrightarrow \mathsf{CatRing}$, and hence upon taking opposites an embedding

$$\mathsf{Aff} := (\mathbb{E}_\infty\mathsf{Ring})^{op} \hookrightarrow \mathsf{Gest}$$

Nota: For $R \in \mathsf{CatRing}$ we write $\mathsf{Spec}(R)$ for the corresponding Gestalt, and for $X \in \mathsf{Gest}$ we write $\underline{\mathsf{Sh}}(X) = (\mathsf{Sh}(X), 2\mathsf{Sh}(X), 3\mathsf{Sh}(X), \dots)$ for the associated categorical ring.

Taking left Kan extensions, the embedding $\mathsf{Aff} \hookrightarrow \mathsf{Gest}$ extends to a functor denoted by $[-]_{\mathsf{QCoh}}$:

$$\begin{tikzcd}
\mathsf{Aff} \arrow[rr, hook] \arrow[dr, hook] & \arrow[d, phantom, "\Downarrow"] & \mathsf{Gest} \\
& \mathsf{PSh}(\mathsf{Aff}) \arrow[ur, dashed, "\mathrm{Lan}=[-]_{\mathsf{QCoh}}"'] &
\end{tikzcd}$$

In particular $n\mathsf{Sh}([X]_{\mathsf{QCoh}}) = n\mathsf{QCoh}(X)$.

- **Key Point:** The functor $[-]_{\mathsf{QCoh}}$ factors through a number of localizations through sheaves with respect to different topologies, including the *fppf* topology and the *descentable* topology.

Action on Qcqs Schemes

If X is a qcqs scheme (classical or derived),

$$[X]_{\mathsf{QCoh}} = \mathsf{Spec}(\mathsf{QCoh}(X))$$

by 1-affineness

- Here 1-affineness is the statement that the co-unit $\mathsf{QCoh}(X) \to \underline{\mathsf{QCoh}}(X)$ is an equivalence, where as a categorical ring

$$\underline{\mathsf{QCoh}}(X) = (\mathsf{QCoh}(X), \mathsf{Mod}_{\mathsf{QCoh}(X)}, \mathsf{Mod}_{\mathsf{Mod}_{\mathsf{QCoh}(X)}}, \dots)$$

- The classical theorem is about the map $\mathsf{Mod}_{\mathsf{QCoh}(X)} \to 2\mathsf{QCoh}(X)$ being an equivalence, while this version also gives equivalences on higher categories.

Because of this,

$$\mathsf{Hom}([X]_{\mathsf{QCoh}}, [Y]_{\mathsf{QCoh}}) = \mathsf{Hom}(\mathsf{QCoh}(Y), \mathsf{QCoh}(X)) \simeq \mathsf{Hom}(X, Y)$$

giving fully-faithfulness of $[-]_{\mathsf{QCoh}}$ on qcqs schemes.

Field Points in $[B\mathbb{G}_m]_{\mathsf{QCoh}}$

Let k be a field and consider

$$\mathsf{Hom}_{\mathsf{Gest}/\mathsf{Spec}(k)}(\mathsf{Spec}(k), [B\mathbb{G}_m]_{\mathsf{QCoh}}) \simeq \mathsf{Hom}(\mathsf{QCoh}(B\mathbb{G}_m), \mathsf{Mod}_k)$$

by a 1-affineness result for $B\mathbb{G}_m$ (this is *not* generally true for algebraic groups).

- **Warning:** Tannaka duality doesn't work outright in this situation

Recall that $\mathsf{QCoh}(B\mathbb{G}_m) \simeq \mathsf{Mod_k}^{\mathbb{Z}}$. Taking π_0,

$$\begin{aligned}\pi_0\mathsf{Hom}_{\mathsf{Gest}/\mathsf{Spec}(k)}(\mathsf{Spec}(k), [B\mathbb{G}_m]_{\mathsf{QCoh}}) &\cong \pi_0\mathsf{Hom}(\mathsf{QCoh}(B\mathbb{G}_m), \mathsf{Mod}_k) \\ &\cong \mathbb{Z} \text{ spanned by } \mathsf{SHEAR}\end{aligned}$$

where $\mathsf{SHEAR}(\bigoplus_i V_i(i)) = \bigoplus_i V_i[2i]$.

Algebraically Closed Field Points in $[B^2\mathbb{Z}/m]_{\mathsf{QCoh}}$

Let k be an algebraically closed field. Then

$$\pi_0\mathsf{Hom}(\mathsf{Spec}(k), [B^2(\mathbb{Z}/m)]_{\mathsf{QCoh}}) \cong \mathbb{Z}/m$$

generated by a $\mathbb{Z}/m$-gerb over k, as appearing as the fiber below:

$$\begin{tikzcd}
\mathrm{Spec}\left(\mathrm{Mod}_k\left[\sqrt[m]{k[2]}\right]\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{pt} \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] & {[B^2(\mathbb{Z}/m)]_{\mathrm{QCoh}}}
\end{tikzcd}$$

We can describe versions of *etale maps* of Gestalten in terms of the categorical concept of adjointability.

n-fold left adjointability

Let $f : X \to Y$ be an m-cell in an (∞, ∞)-category. We say that f is **n-fold left adjointable** if

- **($n = 1$):** f is left adjointable, which is to say it admits a right adjoint in the underlying $(\infty, 2)$-category of the (∞, ∞)-category $\mathrm{Mor}(X, Y)$
- **($n \geq 2$):** f is right adjointable, and the unit and co-unit $(m+1)$-cells are $(n-1)$-fold left adjointable

n-etale Maps of Gestalten

Let $f : X \to Y$ be a map of Gestalten. We say that f is **n-etale** if for all $i \geq 1$, the functor

$$(n+i)\mathrm{Sh}(Y) \to (n+i)\mathrm{Sh}(X)$$

is **i-fold left adjointable** as a map in the $(\infty, n+i+1)$-category of $(\infty, n+i)$-categories.

Note: The left adjointability also encodes some finite presentability hypotheses (e.g. for $\mathrm{Mod}_R \to \mathrm{Mod}_S$ this tells us that S is dualizable over R, and hence finitely presentable).

For $Y \in \mathrm{Gest}$, we have a filtration

$$\mathrm{Gest}^{0\text{-et}}_{/Y} \subseteq \mathrm{Gest}^{1\text{-et}}_{/Y} \subseteq \mathrm{Gest}^{2\text{-et}}_{/Y} \subseteq \cdots \subseteq \mathrm{Gest}_{/Y}$$

though this need not converge. In general, the full subcategory spanned by those $[X]_{\mathrm{QCoh}}$ which are 1-etale over $\mathrm{Spec}(\mathbb{S})$ is closed under colimits and finite limits. Further, the product of two n-etale maps is an $(n+1)$-etale map.

Scholze-Stefanich

$\mathrm{Gest}^{n\text{-et}}_{/Y}$ is (up to local presentability) an $(\infty, 1)$-topos (i.e. it is an *infinitary pre-topos*).

Key Consequence: The functor $X \mapsto [X]_{\mathrm{QCoh}}$ is cocontinuous and *left exact* (though not necessarily a left adjoint)

- **Note:** As of now, it is not known whether $\mathrm{Gest}^{n\text{-}et}_{/Y}$ comes from a site.

Question

Is $\mathrm{Gest}^{n\text{-et}}_{/Y}$ locally small?

- It is not currently known or clear how to prove whether this is locally small.

Transmutation

Goal

We want to build Gestalten out of *six-functor formalisms*.

Let's first recall the necessary details to talk about six-functor formalisms in our current setup:

Correspondences/Spans

Let $\mathcal{C}$ be an $(\infty, 1)$-category with finite limits. Then $\mathrm{Corr}(\mathcal{C})$ or $\mathrm{Span}(\mathcal{C})$ is the symmetric monoidal $(\infty, 1)$-category with the same objects as $\mathcal{C}$, and with

$$\mathrm{Map}_{\mathrm{Corr}(\mathcal{C})}(X, Y) = \{X\} \times_{\mathcal{C}} \mathrm{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{C})^{\simeq} \times_{\mathcal{C}} \{Y\}$$

where the symmetric monoidal structure given by the product in $\mathcal{C}$.

We can now define six-functor formalisms in terms of correspondences.

Six-functor formalism

A **six-functor formalism** in $\mathcal{C}$ is a lax $\otimes$-functor

$$\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \widehat{\mathrm{Pr}}$$

- **Note:** Technically we need to require certain right adjoints, which would exist automatically if Sh factors through Pr.

Claim/Construction (Scholze-Stefanich)

Let $\mathcal{C}$ be an $(\infty,1)$-category with finite limits and let $\mathsf{Sh} : \mathsf{Corr}(\mathcal{C}) \to \widehat{\mathsf{Pr}}$ be a six-functor formalism. Then there is a pullback preserving functor $\mathcal{C} \to \mathsf{Gest}$, $X \mapsto [X]_{\mathsf{Sh}}$, where

$$\mathsf{Sh}([X]_{\mathsf{Sh}}) = \mathsf{Sh}(X)$$

We refer to $[X]_{\mathsf{Sh}}$ as the **transmutation** of X.

- **Naive Idea:** We could compose $\mathcal{C} \hookrightarrow \mathsf{Corr}(\mathcal{C}) \xrightarrow{\mathsf{Sh}} \widehat{\mathsf{Pr}} \xrightarrow{\mathsf{Spec}} \mathsf{Gest}$, but this may not be pullback preserving. We would need a categorical Kunneth formula.

Instead, we should have

$$[X]_{\mathsf{Sh}} = \mathsf{Spec}(\mathsf{Sh}(X), 2\mathsf{Sh}(X), 3\mathsf{Sh}(X), \dots)$$

where we define the higher $n\mathsf{Sh}(X)$ as follows:

- For simplicity assume $\mathsf{Corr}(\mathcal{C}) \xrightarrow{\mathsf{Sh}} \widehat{\mathsf{Pr}}$ is symmetric monoidal, and factors through Pr. For $X \in \mathcal{C}$, we can Kan extend to obtain $\mathsf{PSh}(\mathsf{Corr}(\mathcal{C})) \to \mathsf{Pr}$. Then we define $2\mathsf{Sh}(X)$ as a pushout in $\mathsf{CAlg}(2\widehat{\mathsf{Pr}})$:

$$\begin{tikzcd}
\mathsf{PSh}(\mathsf{Corr}(\mathcal{C})) \arrow[r] \arrow[d, "- \times X"'] \arrow[dr, phantom, "\ulcorner."] & \mathsf{Pr} \arrow[d] \\
\mathsf{PSh}(\mathsf{Corr}(\mathcal{C}_{/X})) \arrow[r] & 2\mathsf{Sh}(X)
\end{tikzcd}$$

More concretely, $2\mathsf{Sh}(X)$ is generated by $\mathsf{Sh}(Y \mid X)$, $Y \in \mathcal{C}_{/X}$, where

$$\mathsf{Hom}(\mathsf{Sh}(Y \mid X), \mathsf{Sh}(Y' \mid X)) \simeq \mathsf{Sh}(Y \times_X Y')$$

Note: Any $X \in \mathcal{C}$ is a commutative algebra object in $\mathsf{Corr}(\mathcal{C})$, with unit and counit

$$X \times X \xleftarrow{\Delta} X = X, \quad \mathsf{pt} \leftarrow X = X$$

so that via a lax symmetric monoidal functor we get commutative algebra objects in $\widehat{\mathsf{Pr}}$.

Question

How can we construct $n\mathsf{Sh}(X)$ in general?

Key assumption: Assume that the functor Sh is symmetric monoidal and factors through Pr moving forward.

$\underline{\mathsf{Corr}}(-)$

For an $(\infty, 1)$-category with finite limits, $\mathcal{C}$, we define the categorical spectrum $\underline{\mathsf{Corr}}(\mathcal{C})$ such that $n\mathsf{Corr}(\mathcal{C})$ is defined as having the same objects as $\mathcal{C}$, and where for $X, Y \in \mathcal{C}$, the morphism category is inductively defined as

$$\mathsf{Hom}_{n\mathsf{Corr}(\mathcal{C})}(X, Y) := (n-1)\mathsf{Corr}({}_{X\backslash}\mathcal{C}_{/Y})$$

Example

A 2-cell in $2\mathsf{Corr}(\mathcal{C})$ is a *span of spans*:

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & U \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\ & T \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We have a natural map of categorical spectra

$$(\mathsf{Corr}(\mathcal{C}), B\mathsf{Corr}(\mathcal{C}), B^2\mathsf{Corr}(\mathcal{C}), \dots) \to \underline{\mathsf{Corr}}(\mathcal{C})$$

Using the adjunction

$$\begin{tikzcd} \mathsf{CatSp} \arrow[r, bend left, "\mathsf{PSh}(-)"] \arrow[r, phantom, "\perp"] & \mathsf{CatRing} \arrow[l, bend left, "\mathsf{forget}"] \end{tikzcd}$$

we then form the following pushout in $\mathsf{CatRing}$ to construct the transmutation $[X]_{\mathsf{Sh}}$:

$$\begin{tikzcd}
\mathsf{PSh}(\mathsf{Corr}(\mathcal{C})) \arrow[rr, "\mathsf{Sh}"] \arrow[d, "- \times X"'] & & \mathsf{Pr} \arrow[dd] \\
\mathsf{PSh}(\mathsf{Corr}(\mathcal{C}_{/X})) \arrow[d] \arrow[r, phantom, "\ulcorner."] & & \\
\mathsf{PSh}(\underline{\mathsf{Corr}}(\mathcal{C}_{/X})) \arrow[rr] & & (2\mathsf{Sh}(X), 3\mathsf{Sh}(X), \ldots)
\end{tikzcd}$$

Generation of $n\mathsf{Sh}(X)$

More concretely, $n\mathsf{Sh}(X)$ is generated by objects $(n-1)\mathsf{Sh}(Y \mid X)$, for $Y \in \mathcal{C}_{/X}$, with hom categories

$$\mathsf{Hom}_{n\mathsf{Sh}(X)}((n-1)\mathsf{Sh}(Y \mid X), (n-1)\mathsf{Sh}(Y' \mid X)) \simeq (n-1)\mathsf{Sh}(Y \times_X Y')$$

- **Intuition:** When $n = 2$ we can think of this as a locally presentable category of kernels.

This hom computation takes place in $\mathsf{PSh}(\mathsf{Corr}(\mathcal{C}_{/X})) \to \mathsf{PSh}(\underline{\mathsf{Corr}}(\mathcal{C}_{/X}))$ using adjunction arguments.

Note: To work with the case where Sh is only lax symmetric monoidal we can use enriched category theory to perform the construction of $[-]_{\mathsf{Sh}}$.

Question

Why does the functor $[-]_{\mathsf{Sh}}$ constructed above preserve pullbacks?

- **Note:** This fact importantly relies on the fact we are working with (∞, ∞)-categories and not (∞, n)-categories for some finite n.

Key Point: In $\mathsf{CatRing}$ we have an equivalence

$$\underline{\mathsf{Sh}}(Y \times_X Y') \simeq \underline{\mathsf{Sh}}(Y) \otimes_{\underline{\mathsf{Sh}}(X)} \underline{\mathsf{Sh}}(Y')$$

where the right hand side is a pushout in $\mathsf{CatRing}$.

Warning

The above formula does **not** hold levelwise - if it did this would be a kind of *relative Kunneth formula*.

However, we do get a levelwise formula if we work *internally over X*, which is to say we get an equivalence

$$n\mathsf{Sh}(Y \times_X Y') \simeq n\mathsf{Sh}(Y|X) \otimes n\mathsf{Sh}(Y' \mid X)$$

in $(n+1)\mathsf{Sh}(X)$.

- **Key Idea:** We can fix Kunneth formulas for n-sheaves by going to $(n+1)$-sheaves

Moduli of Invertible TQFTs

Throughout let R be a categorical ring.

> $n\mathsf{Pic}$
>
> For a categorical ring R, we define $n\mathsf{Pic}(R)$ to be the subcategory of R_n spanned by the $\otimes$-invertible objects. This is equivalent to the category
>
> $$\mathsf{Hom}_{\mathsf{CatRing}}(n\mathsf{PSh}(\mathbb{S}), R)$$
>
> where $n\mathsf{PSh}(\mathbb{S}) = \mathsf{Fun}(\mathbb{S}, (n-1)\mathsf{Pr})$ is the universal categorical ring with a tensor invertible object.

Dually, if $X \in \mathsf{Gest}$,

$$n\mathsf{Pic}(\underline{\mathsf{Sh}}(X)) = \mathsf{Hom}_{\mathsf{Gest}}(X, \mathsf{Spec}(n\mathsf{PSh}(\mathbb{S})))$$

where on the left we have intertible n-sheaves on X, which we think of as *families of invertible n-dimensional TQFTs* over X, so that $\mathsf{Spec}(n\mathsf{PSh}(\mathbb{S}))$ is the carrier of the *universal* such family.

- Due to this, we refer to $n\mathsf{TQFT}_{\mathsf{inv}} := \mathsf{Spec}(n\mathsf{PSh}(\mathbb{S}))$ as the *moduli of invertible n-dimensional TQFTs*.

> **Loops for Moduli of Invertible TQFTs**
>
> For each n, $\Omega(n+1)\mathsf{TQFT}_{\mathsf{inv}} \simeq n\mathsf{TQFT}_{\mathsf{inv}}$.

Nota: We write GL_1 for the resulting *spectrum Gestalt*, which is *1-etale*, and which satisfies

$$\tau_{\geq 0}(\mathsf{GL}_1[n]) = n\mathsf{TQFT}_{\mathsf{inv}}, \quad n \geq 1$$

while $\tau_{\geq 0}\mathsf{GL}_1$ is the *moduli of invertible numbers*, which over a base k is $\mathsf{Spec}(k[\mathbb{S}])$.

> **Stable Homotopy Groups via invertible TQFTs (Scholze-Stefanich)**
>
> Over $\mathbb{C}$, for any $n \geq 1$, we have an equivalence of Gestalten
>
> $$\pi_0(n\mathsf{TQFT}_{\mathsf{inv}}) =: \pi_{-n}(\mathsf{GL}_1) \simeq \mathsf{Hom}(\pi_n(\mathbb{S}), \mathbb{C}^\times)$$

where $\pi_{-n}(\mathsf{GL}_1)$ is a sheaf of abelian groups on Gestalten.

- **Key Point:** We only need to have all roots of unity, and so can replace $\mathbb{C}$ by any characteristic 0 field with all roots of unity.

Example

$\pi_0\mathsf{Hom}(\mathsf{Spec}(\mathbb{C}), \mathsf{GL}_1[1]) \simeq \mathbb{Z}$, but $\pi_1(\mathbb{S}) = \mathbb{Z}/2$. The map $\mathbb{Z} \to \mathbb{Z}/2$ measures how far a sheaf can be locally trivialized in Gestalten.

Interpretation of Theorem in terms of Affine Line

Consider the tautological functor $\underline{\mathsf{Sh}} : \mathsf{Gest}^{op} \to \mathsf{CatRing}$, which is a categorical ring Gestalt, that we denote by $\underline{\mathbb{A}^1}$. The theorem then says we have an equivalence of Gestalten over $\mathbb{C}$

$$\pi_0\, n\mathsf{Pic}(\underline{\mathbb{A}^1}) \simeq \mathsf{Hom}(\pi_n(\mathbb{S}), \mathbb{C}^\times)$$

which suggests that $\underline{\mathbb{A}^1}$ is the *universal target for TQFTs*.

Ind-Coherent Sheaves

The functor $\mathsf{Sh} = \mathsf{IndCoh}$ can be viewed as a six-functor formalism on schemes of finite type over $\mathbb{C}$, which differs from QCoh by some singularities. If X is a smooth scheme of finite type over $\mathbb{C}$, what is $2\mathsf{IndCoh}(X)$? We have $2\mathsf{QCoh}(X) \subseteq 2\mathsf{IndCoh}(X)$, and there exists a 3D TQFT $\mathcal{X}$ with

$$\mathcal{X}(\mathsf{pt}) = 2\mathsf{IndCoh}(X \mid \mathsf{Spec}(\mathbb{C}))$$

and

$$\mathcal{X}(S^1) = \mathsf{IndCoh}(\mathsf{Map}(|S^1|, X)) \simeq \mathsf{QCoh}(T^*[2]X)$$

- **Key Point:** Dualizability, except at the top level, holds more generally due to the fact that n-sheaves admit a map from n-correspondences. However, to get dualizability at the top level, which is needed to define a TQFT (via the cobordism hypothesis), we need to add extra assumptions, such as the smoothness of X above.

This is compatible with the **Rozansky-Witten** theory of T^*X. If there is such a connection, we expect symplectic symmetric of T^*X to act on this TQFT.

(Ben-Zvi, Nadler, Stefanich)

There exists a categorical ring $k(u)_{\mathsf{add}}$, $|u| = 2$, such that for any V, a finite dimensional vector space,

$$\mathsf{2IndCoh}(V) \otimes k(u)_{\mathsf{add}} \simeq \mathsf{2IndCoh}(V^*) \otimes k(u)_{\mathsf{add}}$$

Intuition: This is essentially saying we have a Fourier transform corresponding geometrically to $T^*V \cong T^*V^*$ corresponding to rotations in cotangent spaces.

Nota: If X is a scheme, write $X^! = \mathsf{Spec}(3\mathsf{IndCoh}(X))$.

Shrief of Duals in terms of $B^2\mathbb{G}_m$

For V any finite dimensional vector space, then over $k(u)_{\mathsf{add}}$

$$(V^*)^! = \underline{\mathsf{Hom}}(V^!, B^2\mathbb{G}_m)$$

where the right hand side is the internal hom in the topos of Gestalten.

As a consequence

$$(\mathbb{G}_a)^! \simeq \underline{\mathsf{Hom}}(\mathbb{G}_a^!, B^2\mathbb{G}_m)$$

and $1 \in (\mathbb{G}_a)^!$ corresponds to the exponential map $\mathsf{exp} : \mathbb{G}_a^! \to B^2\mathbb{G}_m$.

Additive de Rham Stack

The additive de Rham stack of $B\mathbb{G}_m$ is

$$(B\mathbb{G}_m)_{\mathsf{dR,add}} = \mathsf{fib}(\mathbb{G}_a^! \xrightarrow{\mathsf{exp}} B^2\mathbb{G}_m)$$

and we have a sequence

$$(B\mathbb{G}_m)_{\mathsf{dR}} \hookrightarrow (B\mathbb{G}_m)_{\mathsf{dR,add}} \to (\mathbb{G}_a)_{\mathsf{dR}}$$

In general, we have Glo, which is a category whose objects consist of quotient stacks $\mathsf{Spec}(A)/G$, where G is reductive. Then there exists a lex functor

$$(-)_{\mathsf{dR,add}} : \mathsf{Glo} \to \mathsf{Gest}_{/k(u)_{\mathsf{add}}}$$

and there also exists multiplicative and elliptic analogues corresponding to various choices of equivariant cohomology theories.

Conjecture (Ben-Zvi, Nadler, Stefanich)

$\mathsf{2Sh}((BG)_{\mathsf{dR},\mathbb{A}^0/\mathbb{Z}}$ satisfies a version of Langlands duality in the sense that

$$\mathsf{2Sh}((BG)_{\mathsf{dR},\mathbb{A}^0/\mathbb{Z}} \simeq \mathsf{2Sh}((BG^\vee)_{\mathsf{dR},\mathbb{A}^0/\mathbb{Z}}$$

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