Categorical Spectra

(March 16-17, Ea E T)

Introduction

This note follows a pair of talks by Naruki Masuda on *categorical spectra*[1] at the FRG workshop on higher categories and geometry.

Tab 1. Table of analogies:

	Classical	Derived	Categorical
Soul:	Set, $(0,0)$-categories	Homotopy types, Ani, $(\infty,0)$-categories	(∞,∞)-categories
Monoidal Structure:	$\times$	$\times$	$\boxtimes$
Enriched Category:	$(1,1)$-category	$(\infty,1)$-category	Oriented category (aka Gray category)
Commutative Grp Objects:	Abelian groups	Grouplike $\mathbb{E}_\infty$-monoids	Symmetric monoidal (∞,∞)-categories, categorical spectra
Base Ring:	$\mathbb{Z}$	$\mathbb{S}$	$\mathbb{F} = B^\infty \mathsf{Fin}^{\simeq}$
Stability:	Abelian Categories	Stable $(\infty,1)$-categories	Stable oriented categories

Categorical Spectra

We now move to defining *categorical spectra*.

Categorical Spectra

The $(\infty,1)$-category of **categorical spectra** is the limit

$$\mathsf{CatSp} := (\infty,\infty)\mathsf{Sp} := \lim\left(\cdots \xrightarrow{\Omega} (\infty,\infty)\mathsf{Cat}_* \xrightarrow{\Omega} (\infty,\infty)\mathsf{Cat}_*\right)$$

in Pr^R, where $(x : \mathsf{pt} \to X)$ is sent by Ω to $(\mathsf{refl}_x : \mathsf{pt} \to \mathsf{End}_X(x))$, where $\mathsf{End}_X(x)$ has a natural $\mathbb{E}_1$-monoidal structure.

Note: Informally, we can think of objects in CatSp as sequences of pointed (∞,∞)-categories $(x_n : \mathrm{pt} \to X_n)_{n\geq 0}$ together with equivalences $(X_n, x_n) \to (\mathrm{End}_{X_{n+1}}(x_{n+1}), \mathrm{refl}_{x_{n+1}})$. An *Eckmann-Hilton argument* can be used to show we have an equivalence

$$\mathsf{CatSp} \simeq \lim\left(\cdots \xrightarrow{\Omega} \mathsf{CMon}((\infty,\infty)\mathsf{Cat}) \xrightarrow{\Omega} \mathsf{CMon}((\infty,\infty)\mathsf{Cat}) \xrightarrow{\Omega} \mathsf{CMon}((\infty,\infty)\mathsf{Cat})\right)$$

Nota: For $-\infty \leq k \leq \infty$, we define $(\infty,k)\mathsf{Sp} \subseteq (\infty,\infty)\mathsf{Sp}$ as the full subcategory spanned by those $(X_n)_{n\geq 0}$ such that X_n is a $\max(n+k,0)$-category. For example,

$$\mathsf{Sp}(\mathsf{Ani}) \simeq (\infty,-\infty)\mathsf{Sp}$$

Example

Many interesting example appear in $(\infty,0)\mathsf{Sp}$ and $(\infty,1)\mathsf{Sp}$.

We can also realize $(\infty,\infty)\mathsf{Sp}$ as the colimit in Pr^L of the sequence

$$\cdots \xleftarrow{\Sigma} (\infty,\infty)\mathsf{Cat}_* \xleftarrow{\Sigma} (\infty,\infty)\mathsf{Cat}_*$$

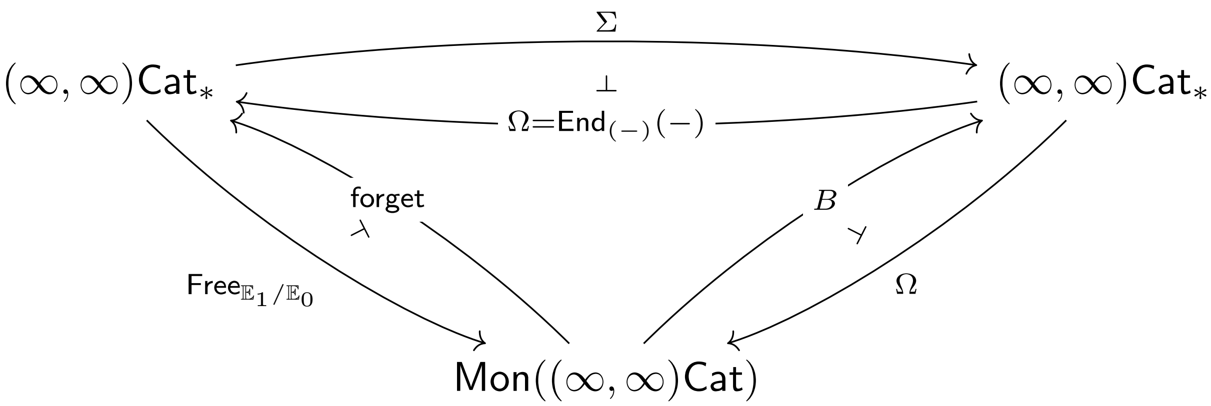
where $\Sigma : (\infty,\infty)\mathsf{Cat}_* \to (\infty,\infty)\mathsf{Cat}_*$ is given by the cofiber

$$\Sigma X := S(X)/S(\mathrm{pt})$$

Further, the right adjoint Ω factors as

$$(\infty,\infty)\mathsf{Cat}_* \xrightarrow{\Omega} \mathsf{Mon}((\infty,\infty)\mathsf{Cat}) \xrightarrow{\text{forget}} (\infty,\infty)\mathsf{Cat}$$

Further, we have a diagram of adjoints:



Remark: Symmetric monoidal (∞,∞)-categories

If we take the colimit in Pr^L of the sequence

$$\cdots \xleftarrow{B} \mathsf{CMon}((\infty,\infty)\mathsf{Cat}) \xleftarrow{B} \mathsf{CMon}((\infty,\infty)\mathsf{Cat})$$

we obtain the $(\infty,1)$-category $(\infty,\infty)\mathsf{SMC}$ which embeds via B^∞ into CatSp. This fits into a pullback diagram

$$\begin{array}{ccc} \mathsf{CMon}^{\mathsf{grp}}(\mathsf{Ani}) & \longrightarrow & (\infty,\infty)\mathsf{SMC} \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{Sp}(\mathsf{Ani}) = (\infty,-\infty)\mathsf{Sp} & \hookrightarrow & \mathsf{CatSp} \end{array}$$

Suspensions and $\boxtimes^{\mathsf{oplax}}$

Note: Throughout $\boxtimes$ will denote the *oplax* Gray tensor product.

Pushouts for Suspensions

We have pushout diagrams

$$\begin{array}{ccc} X \boxtimes \partial \mathsf{D}^1 & \longrightarrow & X \boxtimes \mathsf{D}^1 \\ \downarrow & \ulcorner & \downarrow \\ \partial \mathsf{D}^1 & \longrightarrow & SX \end{array} \qquad \begin{array}{ccc} \partial \mathsf{D}^1 \boxtimes X^{\mathsf{co}} & \longrightarrow & \mathsf{D}^1 \boxtimes X^{\mathsf{co}} \\ \downarrow & \ulcorner & \downarrow \\ \partial \mathsf{D}^1 & \longrightarrow & (SX)^{\mathsf{co}} \end{array}$$

In particular, we have the pushout

$$\begin{array}{ccc} \partial \mathsf{D}^1 \boxtimes X & \longrightarrow & \mathsf{D}^1 \boxtimes X \\ \downarrow & \ulcorner & \downarrow \\ \partial \mathsf{D}^1 & \longrightarrow & (S(X^{\mathsf{co}}))^{\mathsf{co}} = S(X^{\mathsf{coop}}) \end{array}$$

Observation About Suspensions

The functor $S : (\infty,\infty)\mathsf{Cat} \to (\infty,\infty)\mathsf{Cat}_{\partial \mathsf{D}^1/}$ is a map in $\mathsf{LMod}_{(\infty,\infty)\mathsf{Cat}}(\mathsf{Pr}^L)$

- **Key Technical Point:** This functor upgrades to a map of *bimodules*, ${}_{(\infty,\infty)\mathsf{Cat}}\mathsf{BMod}_{(\infty,\infty)\mathsf{Cat}}(\mathsf{Pr}^L)$ up to a *twist*.
  - We have $(\infty,\infty)\mathsf{Cat}^{\mathsf{cop}} \in {}_{(\infty,\infty)\mathsf{Cat}}\mathsf{BMod}_{(\infty,\infty)\mathsf{Cat}}(\mathsf{Pr}^L)$ which has underlying $(\infty,1)$-category $(\infty,\infty)\mathsf{Cat}$, with natural left action, and where the right action is *twisted* by $(-)^{\mathsf{coop}} : (\infty,\infty)\mathsf{Cat} \to (\infty,\infty)\mathsf{Cat}$
    - That is, (A,B) acts on X by $A \boxtimes X \boxtimes B^{\mathsf{coop}}$

Bimodule Functor Extension of S

The functor S admits a unique (up to contractible choice) extension $S : (\infty,\infty)\mathsf{Cat} \to (\infty,\infty)\mathsf{Cat}^{\mathsf{cop}}_{\partial \mathsf{D}^1/}$ to a map in ${}_{(\infty,\infty)\mathsf{Cat}}\mathsf{BMod}_{(\infty,\infty)\mathsf{Cat}}(\mathsf{Pr}^L)$.

⊞ Bimodule Functor Extension of Σ

The functor Σ admits a unique (up to contractible choice) extension $\Sigma : (\infty,\infty)\mathsf{Cat}_* \to (\infty,\infty)\mathsf{Cat}^{\mathsf{cop}}$ to a ${}_{(\infty,\infty)\mathsf{Cat}_*}\mathsf{BMod}_{(\infty,\infty)\mathsf{Cat}_*}(\mathsf{Pr}^L)$.

Due to these observations, we can redefine categorical spectra as the colimit

$$\mathsf{CatSp} := \mathrm{colim}\left(\cdots \xrightarrow{\Sigma} (\infty,\infty)\mathsf{Cat}_* \xrightarrow{\Sigma} (\infty,\infty)\mathsf{Cat}_*^{\mathsf{cop}} \xrightarrow{\Sigma} (\infty,\infty)\mathsf{Cat}_* \xrightarrow{\Sigma} \cdots\right)$$

in ${}_{(\infty,\infty)\mathsf{Cat}_*}\mathsf{BMod}_{(\infty,\infty)\mathsf{Cat}_*}(\mathsf{Pr}^L)$. We have the Gray smash product

$$X \wedge Y := (X \boxtimes Y)/(X \vee Y)$$

on $(\infty,\infty)\mathsf{Cat}_*$, and we will often consider

$$- \wedge \vec{S}^1 : (\infty,\infty)\mathsf{Cat}_* \to (\infty,\infty)\mathsf{Cat}_*$$

where $\vec{S}^1 = B\mathbb{N} = \Sigma S^0$ is the *oriented circle*, and

$$\vec{S}^2 \in Z((\infty,\infty)\mathsf{Cat}_*^{\wedge})$$

is *central*. Further, $\vec{S}^3$ has a $(1\,2\,3)$ cycle action.

- **Key Point:** This follows from a similar argument as $\mathsf{Ani} \to \mathsf{Sp}(\mathsf{Ani})$ in Pr^{L} being an *idempotent* algebra, and $\mathsf{Sp} \simeq \mathsf{Ani}[(S^1)^{-1}]$. (*Note:* we have a Σ_3 action on S^3)

⊡ Categorical Spectra as a *mode*

$(\infty,\infty)\mathsf{Cat}^{\boxtimes} \xrightarrow{\Sigma^\infty_+} \mathsf{CatSp} \in {}_{(\infty,\infty)\mathsf{Cat}}\mathsf{BMod}_{(\infty,\infty)\mathsf{Cat}}(\mathsf{Pr}^L_\omega)$ is an idempotent algebra, and so there exists a unique (up to contractible choice) tensor product $\otimes$ promoting Σ^∞_+ to a monoidal functor, and

$$\mathsf{CatSp}^{\otimes} \simeq (\infty,\infty)\mathsf{Cat}_*^{\wedge}[(\vec{S}^1)^{-1}]$$

where the inversion $(\vec{S}^1)^{-1}$ is from *both sides*.

Key Point: Categorical spectra is *mode* of (bi-)oriented categories.

- Here a bi-oriented category is a $((\infty,\infty)\mathsf{Cat}^{\boxtimes}, (\infty,\infty)\mathsf{Cat}^{\boxtimes})$-bienriched category (i.e. these are what classically are called *Gray-bimodules*)

> **Question**
>
> What is the center of $((\infty,\infty)\mathsf{Cat}, \boxtimes)$?

The essential image of the forgetful functor

$$_{(\infty,\infty)\mathsf{Sp}}\mathsf{BMod}_{(\infty,\infty)\mathsf{Sp}}(\mathsf{Pr}^L) \hookrightarrow {}_{(\infty,\infty)\mathsf{Cat}}\mathsf{BMod}_{(\infty,\infty)\mathsf{Cat}}(\mathsf{Pr}^L))$$

consists of those bi-oriented categories $\mathcal{C}$ such that $\mathcal{C}$ is *pointed* and $\Sigma \dashv \Omega$ is an adjoint equivalence, or equivalently $\mathcal{C}$ is pointed and the action of $\vec{S}^1$ on $\mathcal{C}$ is invertible.

> **Stable Gray (bi-)module**
>
> A Gray (bi-)module is **stable** if it is pointed and the action by the oriented circle $\vec{S}^1$ is invertible.

> **Question**
>
> - Why is this non-locally presentable?
> - What is a stronger characterization?

> **Digression**
>
> Stable $(\infty,1)$-categories are finitely (co)complete $(\infty,1)$-categories where limits and colimits **commute**
>
> - **Note:** For discrete limits and colimits, i.e. products and coproducts, this is equivalent to them coinciding. For pullbacks and pushouts, this is equivalent to pullback squares being the same as pushout squares.
> - This fits in the general framework of *higher semiadditivity* in a precise sense
>
> We say an $(\infty,1)$-category is "quasi-stable" if it is 0-semiadditive, and has loops and suspensions such that $\Sigma \dashv \Omega$ is an adjoint equivalence (to be stable we need to also include fibers and cofibers). This quasi-stability is still true in CatSp, but the statement that pullback and pushout squares coincide is no longer true in CatSp.
>
> - **Aside:** Oriented pushouts do not paste (though we have a pasting law for oriented pushouts with homotopy/psuedo-pushouts)

The following equivalent characterizations of stability will be important for generalization to categorical spectra.

Exercise

The following are equivalent for $\mathcal{C}$ an $(\infty,1)$-category:

- **(1)** $\mathcal{C}$ is stable
- **(2)** For any finite category J, $\mathcal{C} \xrightarrow{\Delta} \mathrm{Fun}(J,\mathcal{C})$ has a left adjoint colim, and the further left adjoint exists
  - **Key Point:** The colimit also has a mapping in property!!
- **(3)** For $J = \emptyset$, pt $\sqcup$ pt, $\{\bullet \leftarrow \bullet \rightarrow \bullet\}$, **(2)** holds

Left Adjoint to Oriented Pushout

We have an oriented pushout functor

$$\mathrm{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathrm{CatSp}) \xrightarrow{(-)\vec{\sqcup}_{(-)}(-)} \mathrm{CatSp}$$

which admits a left adjoint $\Sigma^{\infty-1}(I^{op} \leftarrow S^0 \rightarrow I) \otimes -$ where I is the walking arrow based at the source of the arrow.

Weighted Colimit Characterizations of Stability

Absolute Weighted Colimits

For $\mathsf{V} \in \mathrm{Alg}(\mathrm{Pr}^L)$, and $\mathcal{C}$ left V-enriched, a weight $W : J^{op} \to \mathsf{V}$ is **absolute** for the V-enrichment of $\mathcal{C}$ if and only if we have a weighted colimit functor

$$W \otimes - := \mathrm{colim}^W : \mathrm{Fun}(J,\mathcal{C}) \to \mathcal{C}$$

and this admits a left adjoint in $\mathrm{RMod}_{\mathsf{V}}(\mathrm{Pr}^L)$.

For $J = \emptyset$, pt $\sqcup$ pt, $(\bullet \leftarrow \bullet \rightarrow \bullet)$, or any finite category, an $(\infty,1)$-category $\mathcal{C}$ is *stable* if and only if each conical weight $W : J^{op} \to \mathrm{An}$ is *absolute* for the An-enrichment of $\mathcal{C}$.

- **Key Idea:** Stability = "finite weight"ed (conical) colimits are absolute

Absolute Span Weight for Categorical Spectra

If $W = (\{0\} \to \mathrm{D}^1 \leftarrow \{1\}) : (\bullet \leftarrow \bullet \rightarrow \bullet)^{op} \to (\infty,\infty)\mathrm{Cat}$ is the cospan for the walking 1-cell, then W is absolute for the $(\infty,\infty)\mathrm{Cat}$-enrichment of CatSp. The W-weighted colimit is given on categorical spectra by

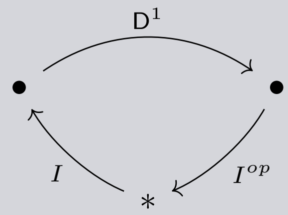
$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & & \\ Y & & \end{array} \quad\mapsto\quad \text{colim}\left(\begin{array}{ccccc} & & X & \longrightarrow & Z \\ & & \downarrow & & \\ X & \longrightarrow & \mathrm{D}^1 \boxtimes X & & \\ \downarrow & & & & \\ Y & & & & \end{array}\right)$$

The left adjoint is given by tensoring with W^L, which is $\Sigma^{\infty-1}(I^{op} \leftarrow S^0 \rightarrow I)$ for $I = (0 \rightarrow 1)$ pointed at 0.

- **Key Idea:** Think of "S-duality" of profunctors weights, $W^L \dashv W$

The unit $\mathbb{F} \rightarrow \text{colim}^W(W^L) = W \otimes W^L = \Sigma^{\infty-1}(L)$, or equivalently $\vec{S}^1 \rightarrow L$ where L is the colimit of the diagram below

$$\begin{array}{ccccc} & & S^0 & \longrightarrow & I \\ & & \downarrow & & \\ S^0 & \longrightarrow & \mathrm{D}^1_+ \wedge S^0 = \mathrm{D}^1_+ & & \\ \downarrow & & & & \\ I^{op} & & & & \end{array} \quad\rightsquigarrow$$



where $*$ denotes the basepoint.

Left Stable

A left (∞,∞)Cat-enriched category $\mathcal{C}$ is said to be **left stable** if it is pointed, has directed pushouts $\vec{\cup}$ (i.e. W-weighted colimits), and W is absolute for the (∞,∞)Cat-enrichment of $\mathcal{C}$.

One can similarly define for bi-enriched.

- **Key Point:** The fact that W is absolute for the (∞,∞)Cat-enrichment of $\mathcal{C}$ can be weakened to the adjunction $\vec{\Sigma} \dashv \vec{\Omega}$ being an adjoint equivalence.

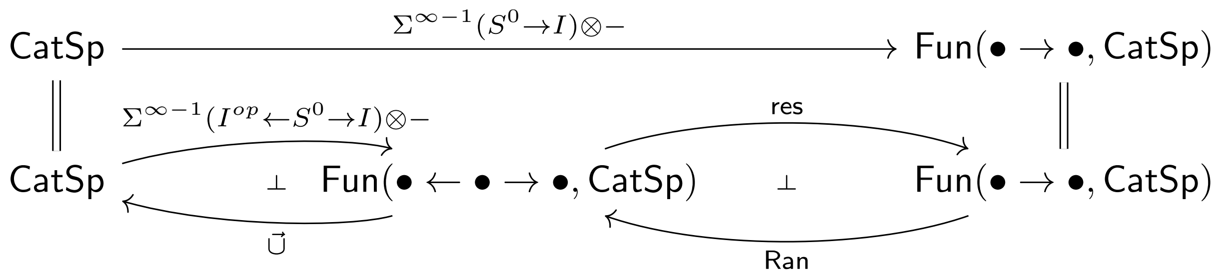
Example

We have an embedding

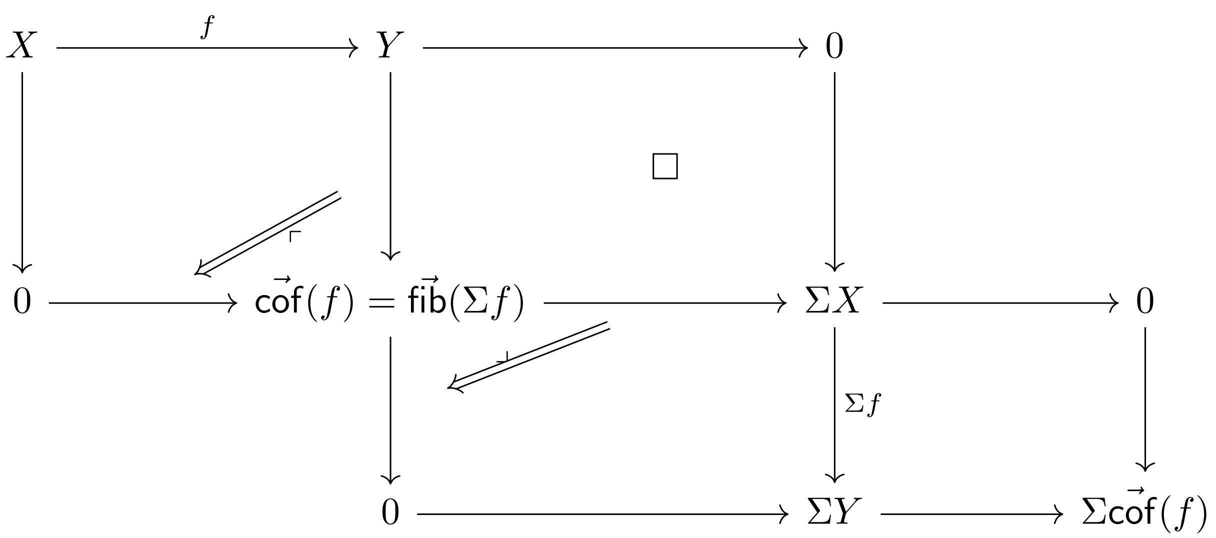
$$(\infty,1)\mathsf{Cat} = \mathsf{An}\text{-}\mathsf{Cat}\text{-}\mathsf{An} \hookrightarrow (\infty,\infty)\mathsf{Cat}^{\boxtimes}\text{-}\mathsf{Cat}\text{-}(\infty,\infty)\mathsf{Cat}^{\boxtimes}$$

which preserves and reflects the notion of *stable $(\infty,1)$-categories*. This follows from the fact that directed pushouts degenerate to ordinary pushouts.

Looking at the adjoints



From here we can construct a **Barratt-Puppe sequence**:



Spanier-Whitehead Duality

For spectra, $J = \mathsf{pt}$, $X =$ a finite spectrum, we have an adjunction $X \otimes - : \mathsf{Sp} \leftrightarrows\ : X^L \otimes -$.

Characterizing dualizable categorical spectra

The following are equivalent for a categorical spectrum X:

- X is **perfect**, i.e. it is generated by $\mathbb{F}, 0, \Sigma^{-1}, \vec{\cup}$, and retracts
- X is **tiny**, i.e. $[X, -] : \mathsf{CatSp} \to \mathsf{CatSp}$ commutes with weighted colimits in CatSp
- X is **dualizable** with respect to $\otimes$

Categorical Spectra with Adjoints

Motivation

Studying higher categories is often motivated by TQFTs (topological quantum field theories), $\otimes$-functors $\mathsf{Bord}_n \to \mathcal{D}$, and six-functor formalisms, lax $\otimes$-functors $(n)\mathsf{Corr}(\mathcal{C}) \to \mathcal{D}$.

Recall: For $\mathcal{C}$ an (∞, ∞)-category, an adjunction in $\mathcal{C}$ is defined to be an adjunction in $\mathsf{Ho}_2(\mathcal{C})$, which is equivalent to a functor $\mathsf{Adj} \to \mathcal{C}$ from the walking adjunction, where Adj is a strict 2-category consisting of the following data:

- Objs: Objects x, y
- 1-Cells: Morphisms generated by $\ell : x \to y$ and $r : y \to x$
- 2-Cells: 2-cells generated by $\eta : \mathrm{id} \to r\ell$ and $\epsilon : \ell r \to \mathrm{id}$, subject to triangle identities

A morphism $\mathsf{D}^1 \to \mathcal{C}$ has a right adjoint if it extends along the inclusion $\mathsf{D}^1 \xrightarrow{\ell} \mathsf{Adj}$.

Riehl-Verity + Folklore

The functor $\mathsf{D}^1 \to \mathsf{Adj}$ is an epimorphism in univalent $(\infty, \infty)\mathsf{Cat}$ (proved for $(\infty, 2)\mathsf{Cat}^{\mathsf{univ}}$).

Left Adjoints for k-Morphisms

An (∞, ∞)-category $\mathcal{C}$ has **left adjoints for k-morphisms** if and only if $\mathcal{C} \to \mathsf{pt}$ is weakly right orthogonal to $S^{k-1}\mathsf{D}^1 \xrightarrow{S^{k-1}\ell} S^{k-1}\mathsf{Adj}$. Let $(\infty, \infty)\mathsf{Cat}^{n\text{-adj}}$ denote the full subcategory of (∞, ∞)-categories with adjoints for $k < n$-morphisms.

For n finite, $(\infty, n)\mathsf{Cat}^{\mathsf{adj}} := (\infty, n)\mathsf{Cat} \cap (\infty, \infty)\mathsf{Cat}^{n\text{-adj}}$.

Adjointable n-connective categorical spectra

We define

$$(\infty, n)\mathsf{Sp}^{\mathsf{adj}} := \lim \left(\cdots \xrightarrow{\Omega} (\infty, n+k)\mathsf{Cat}_*^{\mathsf{adj}} \xrightarrow{\Omega} (\infty, n+k-1)\mathsf{Cat}_*^{\mathsf{adj}} \xrightarrow{\Omega} \cdots \right)$$

Example

- $\{n\mathsf{Corr}(\mathcal{C})\}_n \in (\infty, 0)\mathsf{Sp}^{\mathsf{adj}}$
- For a tangential structure X, $B^{\infty-n}\mathsf{Bord}_n^X \in (\infty, 0)\mathsf{Sp}^{\mathsf{adj}}$

Monoidal Structure on Categorical Spectra descends along adding adjunctions

Asking for adjoints in categorical spectra is a localization, and the tensor product descends over this localization:

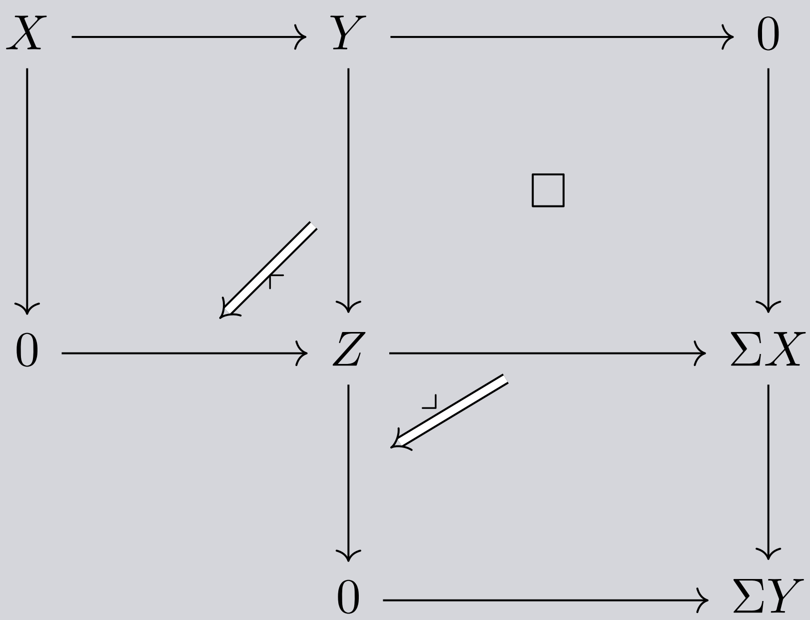
$$\begin{array}{ccc} (\infty,n)\mathsf{Sp}\otimes(\infty,m)\mathsf{Sp} & \xrightarrow{\otimes} & (\infty,n+m)\mathsf{Sp} \\ \downarrow & & \downarrow \\ (\infty,n)\mathsf{Sp}^{\mathsf{adj}}\otimes(\infty,m)\mathsf{Sp}^{\mathsf{adj}} & \xrightarrow[\exists\otimes]{} & (\infty,n+m)\mathsf{Sp}^{\mathsf{adj}} \end{array}$$

In particular, we obtain a localized tensor product on $(\infty,0)\mathsf{Sp}^{\mathsf{adj}}$ and $(\infty,\infty)\mathsf{Sp}^{\mathsf{adj}}$.

Key Idea: Write $\mathsf{D}^1\boxtimes\mathsf{D}^1 \xrightarrow{\mathsf{D}^1\boxtimes\ell} \mathsf{D}^1\boxtimes\mathsf{Adj}$ as a pushout of $\mathsf{D}^1\xrightarrow{\ell}\mathsf{Adj}$ and $S\mathsf{D}^1\xrightarrow{S\ell} S\mathsf{Adj}$, and use *mate calculus*.

Categorical Spectra with Adjoints in (Co)fibers

For any diagram of categorical spectra below:



if $Y,\Sigma X\in(\infty,n)\mathsf{Sp}^{\mathsf{adj}}$, then $Z\in(\infty,n)\mathsf{Sp}^{\mathsf{adj}}$ and Z is closed under extensions.

Using this work we can translate the cobordism hypothesis to this language:

Example

The cobordism hypothesis trasnlates to the statement that we have in the adjunction

$$(\infty,0)\mathsf{Sp}\leftrightarrows(\infty,0)\mathsf{Sp}^{\mathsf{adj}}$$

> S^{-n} is mapped to $B^{\infty-n}\mathsf{Bord}_n^{\mathrm{fr}}$ (and similarly for a general tangential structure). Then the extension $(X.Y \rightsquigarrow Z)$ produces cobordism category with defects. The limit description recovers known cells, while the colimit description recovers the universal property for the *cobordism hypothesis with singularities*.

We have the following diagram of localizations

$$\begin{array}{ccccc} & & (\mathsf{CatSp}, \boxtimes) & & \\ & \swarrow & & \searrow & \\ ((\infty,0)\mathsf{Sp}^{\mathrm{adj}}, \otimes) & & & & ((\infty,\infty)\mathsf{Sp}^{\mathrm{adj}}, \otimes) \\ & \searrow & & \swarrow & \\ & & (\mathsf{Sp}, \otimes) & & \end{array}$$

Conditional Argument:

Recall

$$(\infty,0)\mathsf{Sp}^{\mathrm{adj}} = \operatorname*{colim}_{\mathbb{N}} \Big((\infty,0)\mathsf{Cat}_*^{\mathrm{adj}} \to (\infty,1)\mathsf{Cat}_*^{\mathrm{adj}} \to \cdots \Big)$$

where $(\infty,n)\mathsf{Cat}_*^{\mathrm{adj}}$ has an $O(n)$-action induced by an equivalence,

$$(\infty,n)\mathsf{Cat}_*^{\mathrm{adj}} \simeq \mathsf{Fun}^{\infty\mathrm{exc}}(\mathsf{Mfld}_n^{\mathrm{sfr}}, \mathsf{An})$$

where $\mathsf{Mfld}_n^{\mathrm{sfr}}$ has a natural $O(n)$-action. **If** this diagram lifts to a symmetric monoidal functor

$$\mathsf{FinVect}^{\mathrm{inj}} \to \mathsf{Pr}^L$$

where $\mathbb{N} \to \mathsf{FinVect}^{\mathrm{inj}}, \;\; n \mapsto \mathbb{R}^n$ is final (i.e. colimits over $\mathsf{FinVect}^{\mathrm{inj}}$ coincide with colimits of diagrams restricted to $\mathbb{N}$). From here one would get that

$$(\infty,0)\mathsf{Sp}^{\mathrm{adj}} \simeq \operatorname*{colim}_{\mathsf{FinVect}^{\mathrm{inj}}} ((\infty,n)\mathsf{Cat}_*^{\mathrm{adj}}) \in \mathsf{CAlg}(\mathsf{Pr}^L)$$

> ⚠ **Warning**
>
> Although $\mathbb{N} \to \mathsf{FinVect}^{\mathrm{inj}}$ is monoidal, it is **not** symmetric monoidal. Indeed, $n \in \mathbb{N}$ has a trivial Σ_n-action, while $\mathbb{R}^n$ has a non-trivial Σ_n-action. To get a symmetric monoidal functor we need to first pass to $\mathsf{FinSet}^{\mathrm{inj}} \to \mathsf{FinVect}^{\mathrm{inj}}$

References

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1. Masuda, Naruki. “The Algebra of Categorical Spectra.” Johns Hopkins University, 2024. https://jscholarship.library.jhu.edu/handle/1774.2/70013 . ↩