Towards the Categorification of Homotopy Theory - Gepner

(March 16, Ea E T)

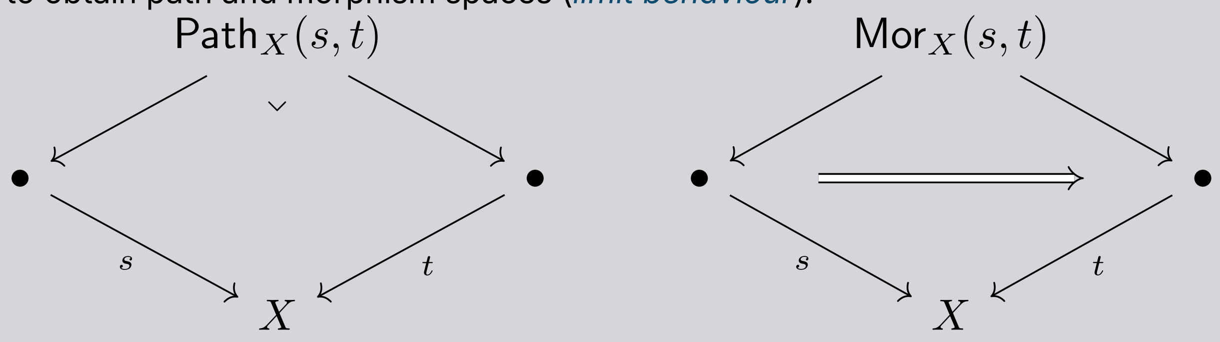
Introduction

This note follows a talk by David Gepner as part of a mini-course with Hadrian Heine on the *Categorification of homotopy theory*[1] at the FRG workshop on higher categories and geometry.

> **Premise**
>
> Lift homotopy theory to higher categorical world.

Intuition: We will think of ∞-categories (i.e. (∞, ∞)-categories) as *generalized spaces*. (here n-category will mean (∞, n)-category). In spaces we can decompose any space X as a gluing of cells.

> **Goal**
>
> - We may want a *skeletal filtration* $\mathsf{sk}_n\, X$ on ∞-categories, like we have in spaces (*colimit behaviour*)
> - We should be able to take (homotopy/pseudo) pullbacks and *oriented* pullbacks to obtain path and morphism spaces (*limit behaviour*):



Key Idea: When doing these constructions we *reduce* the category level. If we want to not eventually end up at spaces, we need to work with ∞-categories (i.e. (∞, ∞)-categories).

- **Note:** We start the base of induction at $\mathcal{S}$, spaces, homotopy types, anima, ∞-groupoids, in which case we are working homotopically.

We obtain a tower of higher notions of categories

$$\cdots \qquad (\infty,2)\mathsf{Cat} \underset{\iota_1}{\overset{\tau_1}{\rightleftarrows}} (\infty,1)\mathsf{Cat} = \mathcal{S}\text{-}\mathsf{Cat} \underset{\iota_0=\mathsf{Core}(-)}{\overset{\tau_0=\mathsf{colim}}{\rightleftarrows}} (\infty,0)\mathsf{Cat} = \mathcal{S}$$

$$(\infty,1)\mathsf{Cat} \subseteq \mathsf{Fun}(\Delta^{op},\mathcal{S})$$

We then define $(\infty,\infty)\mathsf{Cat}$ as the limit over the core functors, ι_j, which forget the higher cells.

$$(\infty,\infty)\mathsf{Cat} := \lim\left(\cdots \xrightarrow{\iota_2} (\infty,2)\mathsf{Cat} \xrightarrow{\iota_1} (\infty,1)\mathsf{Cat} \xrightarrow{\iota_0} (\infty,0)\mathsf{Cat}\right) \in \mathsf{Pr}^R \subseteq (\infty,1)\mathsf{CAT}$$

> **Alternatives**
>
> We could also take the limit along τ_j-functors, though this results in a *different* but *related* notion.

Nota: Map will be referring to mapping spaces in $(\infty,1)$-categories, while Mor will refer to more general mapping (∞,∞)-categories in an (∞,∞)-category.

Categorifying Homotopy Theory (David Gepner)

To begin with categorifying homotopy theory we can define *suspension* functors for higher ∞ -categories.

> **Suspension Functor**
>
> We define the suspension $S : (\infty,n)\mathsf{Cat} \to (\infty,n+1)\mathsf{Cat}$ by sending an (∞,n)-category X to the $(\infty,n+1)$-category with objects 0 and 1, and with $\mathsf{Mor}(0,1) = X$ the (∞,n)-category we started with. In particular,
>
> $$S(X) := 0 \blacktriangleright_X 1$$

In the limit we get $S : (\infty,\infty)\mathsf{Cat} \to (\infty,\infty)\mathsf{Cat}_{\partial[1]/}$, and this has a right adjoint

$$\mathsf{Mor}_{(-)}(-,-) : (\infty,\infty)\mathsf{Cat}_{\partial[1]/} \to (\infty,\infty)\mathsf{Cat}, \quad (X,(s,t)) \mapsto \mathsf{Mor}_X(s,t)$$

> **Disks and Boundaries**
>
> Let $\mathsf{D}^0 = \mathsf{pt}$, and then we inductively define $\mathsf{D}^1 := S(\mathsf{D}^0)$ (the *walking arrow*), $\mathsf{D}^2 := S(\mathsf{D}^1)$ (the *walking 2-cell*), etcetera. with $S(\mathsf{D}^n) =: \mathsf{D}^{n+1}$ the *walking n-cell*.
>
> We define the boundary as removing the top level cells, so that $\partial\mathsf{D}^0 = \emptyset$, $\partial\mathsf{D}^1 := S(\partial\mathsf{D}^0) = \mathsf{pt} \sqcup \mathsf{pt}$, $\partial\mathsf{D}^2 := S(\partial\mathsf{D}^1) = \{0 \rightrightarrows 1\}$, etcetera, with $\partial\mathsf{D}^{n+1} := S(\partial\mathsf{D}^n)$. Applying groupoidification, τ_0, we have
>
> $$\tau_0\partial\mathsf{D}^n \simeq S^{n-1} \hookrightarrow \tau_0\mathsf{D}^n$$

We then have a cofiber sequence

$$\partial \mathsf{D}^n \hookrightarrow \mathsf{D}^n \xrightarrow{\text{cof}} \mathbb{S}^n$$

where $\mathbb{S}^n$ is trivial up to dimension n, and is equivalent to the n-fold *delooping* $B^n\mathsf{Conf}(\mathbb{R}^n)$.

We also have a cofiber sequence

$$\partial \mathsf{D}^n \to \tau_{n-1}\mathsf{D}^n \to \mathsf{S}^n$$

where S^n is the n-dimensional un-oriented sphere. Here $\tau_{n-1}\mathsf{D}^n$ inverts the non-degenerate n-cell, so that $\tau_{n-1}\mathsf{D}^n \simeq \mathsf{D}^{n-1}$.

Goal

We also want to *categorify* the cartesian product

Question

What do we mean by *categorifying* the cartesian product?

Key Idea: Specifically, we want to think of the cartesian product of *spaces* as

$$X \times Y \simeq \coprod_X Y$$

and we want to categorify the right side interpretation of the cartesian product. Doing this gives us instead the **Gray tensor product**.

- **Note:** Combinatorics of Gray tensor products live in chain complexes (c.f. Strict Inf-Cats for more details)

Example

We can think of the walking arrow $\mathsf{D}^1 = \{s \xrightarrow{x} t\}$ as living inside the chain complex $C_\bullet(\mathsf{D}^1)$

$$\begin{array}{ccc} \text{deg } 0 & & \text{deg } 1 \\ \mathbb{Z}\{s,t\} & \xleftarrow[\partial(x)=t-s]{\partial} & \mathbb{Z}\{x\} \end{array}$$

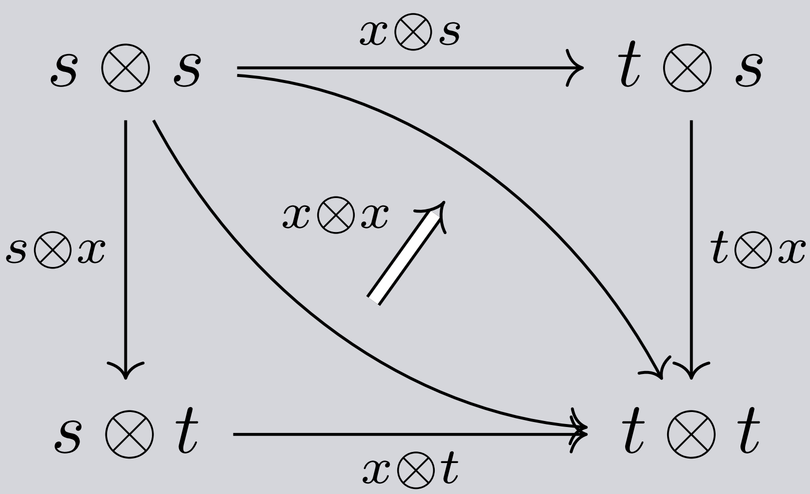
Then we can define $\mathsf{D}^1 \boxtimes \mathsf{D}^1$ as living in $C_\bullet(\mathsf{D}^1) \otimes C_\bullet(\mathsf{D}^1)$ using the usual convention of differentials in $(\mathsf{Ch}(\mathbb{Z}), \otimes)$, so that

$$\partial(x \otimes x) = \partial x \otimes x - x \otimes \partial x = (t-s) \otimes x - x \otimes (t-s) = t \otimes x - s \otimes x - x \otimes t + x$$

where we are interpreting *sums* as compositions, so that as

$$\partial(x \otimes x) = (t \otimes x + x \otimes s) - (s \otimes x + x \otimes t)$$

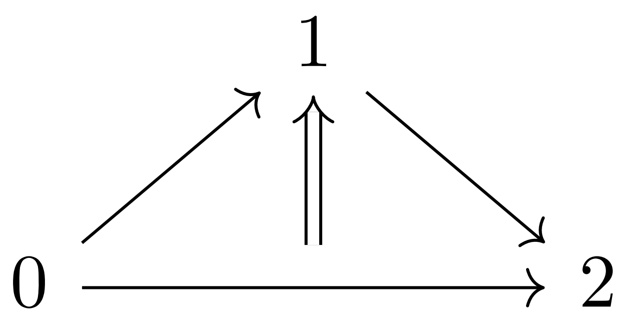
tells us how we should orient our cell $x \otimes x$:



Nota: we define $\square^n := (\mathsf{D}^1)^{\boxtimes n}$, and we write $\square \subseteq (\infty,\infty)\mathsf{Cat}$ for the resulting full subcategory, and get an adjunction of *large* $(\infty,1)$-categories:

$$\left(\mathsf{PSh}(\square), \boxtimes\right) \rightleftarrows \left((\infty,\infty)\mathsf{Cat}, \boxtimes\right) \quad \bot$$

We can similarly work with simplices Δ^n as (∞, n)-categories, with $\Delta^n \subseteq C_\bullet(\Delta^n)$, so that instead of gray tensor products we get the **join**. Note that here Δ^2 is a strict 2-category with orientations:



so that $\Delta^n \star \Delta^m = \Delta^{n+1+m}$, and we get a similar adjunction

$$\left(\mathsf{PSh}(\Delta), \star\right) \rightleftarrows \left((\infty,\infty)\mathsf{Cat}, \star\right) \quad \bot$$

The *join* and *gray tensor* are related by the following pushouts:

$$\begin{array}{ccc} X \boxtimes \partial \mathsf{D}^1 \boxtimes Y & \longrightarrow & X \boxtimes \mathsf{D}^1 \boxtimes Y \\ \downarrow & \ulcorner. & \downarrow \\ X + Y & \longrightarrow & X \star Y \end{array} \qquad \begin{array}{ccc} X \boxtimes \partial \mathsf{D}^1 & \longrightarrow & X \boxtimes \mathsf{D}^1 \\ \downarrow & \ulcorner. & \downarrow \\ \partial \mathsf{D}^1 & \longrightarrow & S(X) \end{array}$$

Left and Right Closed Monoidal Properties of Gray Tensor and Join

For $X, Y, Z \in (\infty, \infty)\mathsf{Cat}$,

$$\mathsf{Map}(X, \mathsf{Fun}^{\mathsf{oplax}}(Y, Z)) := \mathsf{Map}(X \boxtimes Y, Z) =: \mathsf{Map}(Y, \mathsf{Fun}^{\mathsf{lax}}(X, Z))$$

and for any $g : Y \to Z$ and $f : X \to Z$,

$$\mathsf{Map}_{Y/}(X \star Y, Z) =: \mathsf{Map}(X, Z_{//g}), \quad \mathsf{Map}(Y, Z_{f//}) := \mathsf{Map}_{X/}(X \star Y, Z)$$

Oriented Categories

An **oriented category** is a *left-enriched* category, as in $((\infty, \infty)\mathsf{Cat}, \overline{\boxtimes})$-Cat, or equivalently a *right-enriched* category, as in Cat-$((\infty, \infty)\mathsf{Cat}, \boxtimes)$, where

$$X \overline{\boxtimes} Y = (X^{\mathsf{co}} \boxtimes Y^{\mathsf{co}})^{\mathsf{co}}$$

where $(-)^{\mathsf{co}}$ reverses *even dimensional cells* and $(-)^{\mathsf{op}}$ reverses *odd dimensional cells*.

Cylinder

For X an $(\infty, \infty)\mathsf{Cat}$, we define its **cylinder**

$$\mathsf{Cyl}(X) := X \boxtimes \mathsf{D}^1$$

which is an endofunctor of $(\infty, \infty)\mathsf{Cat}$. However this is **not** compatible with the cartesian monoidal enrichment in $((\infty, \infty)\mathsf{Cat}, \times)$. Indeed, taking $X = \mathsf{D}^0$, we have no map

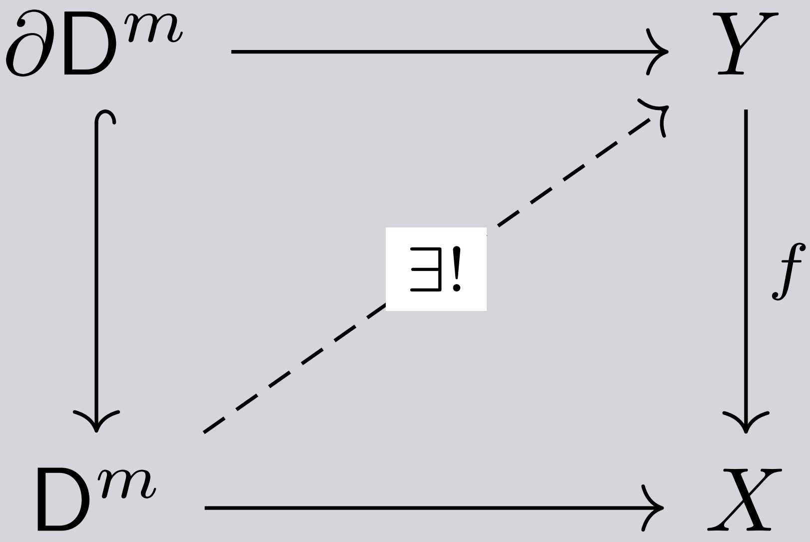
$$\mathsf{D}^1 \times \mathsf{D}^1 = \mathsf{D}^1 \times \mathsf{Cyl}(\mathsf{D}^0) \to \mathsf{Cyl}(\mathsf{D}^1 \times \mathsf{D}^0) = \mathsf{D}^1 \boxtimes \mathsf{D}^1$$

Beginnings of Homotopy Theory

To begin doing (∞, ∞)-categorical homotopy theory, we can define connectedness and truncations.

Truncated and Connected Map

A map $f : Y \to X$ of (∞, ∞)-categories is **n-truncated** if for all $m > n$, any square below has a contractible space of solutions:



n-truncated maps form the right class for an orthogonal factorization system, where the left class is the class of **n-connected** maps.

Notation: We define

$$\mathsf{Fun}^{\mathsf{oplax}}(X, Y) := \mathsf{Mor}_{(\infty,\infty)\mathsf{Cat}^{\mathsf{or}}}(X, Y) \quad \text{and} \quad \mathsf{Fun}(X, Y) := \mathsf{Mor}_{(\infty,\infty)\mathsf{Cat}}(X, Y)$$

We say that an object X of an **oriented category** $\mathcal{C}$ is **n-truncated** if $\mathsf{Mor}_{\mathcal{C}}(Y, X) \in (\infty, \infty)\mathsf{Cat}$ is n-truncated (i.e. its map into the terminal object is n-truncated).

Postnikov Towers

From the notion of n-truncated maps and skeletal filtrations on any (∞, ∞)-category, we obtain the notion of **Postnikov towers** in this context.

- Consider the subcategory $\Delta^+ := \{\Delta^n, \tau_{n-1}\Delta^n\}_{n\in\mathbb{N}} \subseteq (\infty, \infty)\mathsf{Cat}$ with *atomic* maps between them. The *complicial philosophy*, conjectured by Robert Street, says that we can present $(\infty, \infty)\mathsf{Cat}$ as certain pre-sheaves on Δ^+.
- **Key Idea:** This perspective is beneficial on a cellular level, since we can filter Δ^+ by cardinality, which corresponds roughly to dimension, and then we get a filtration $X = \mathsf{colim}_n\, \mathsf{sk}_n X$ where

$$\mathsf{sk}_n X = \underset{\Delta^+_{\leq n}\downarrow X}{\mathsf{colim}}\ \Delta^m$$

  is the best *colimit* approximation of X by $\Delta^+_{\leq n}$. Specifically, we have a pushout where we index over *non-degenerate* simplices

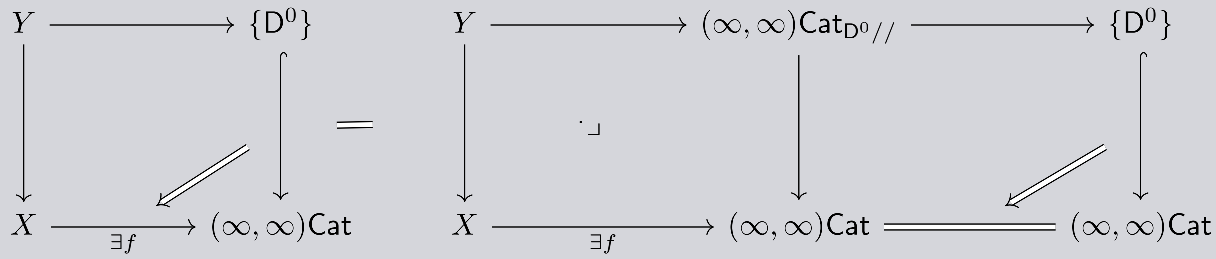
$$\begin{array}{ccc}
\coprod_{\tau_{n-1}\Delta^n \to X} \partial \mathsf{D}^n + \coprod_{\Delta^n \to X} \partial \mathsf{D}^n & \longrightarrow & \mathsf{sk}_{n-1} X \\
\big\downarrow & \ulcorner . & \big\downarrow \\
\coprod_{\tau_{n-1}\Delta^n \to X} \tau_{n-1} \mathsf{D}^n + \coprod_{\Delta^n \to X} \mathsf{D}^n & \longrightarrow & \mathsf{sk}_n X
\end{array}$$

Key Point: These filtrations allow us to define an *obstruction theory* of (∞,∞)-categories.

$$\mathsf{sk}_n X / \mathsf{sk}_{n-1} X = (\bigvee S^n) \vee (\bigvee \mathbb{S}^n)$$

Cocartesian Fibrations

A morphism $p : Y \to X$ in $(\infty,\infty)\mathsf{Cat}$ is a **cocartesian fibration** if it is classified:



References

1. Gepner, David, and Hadrian Heine. “Oriented Category Theory.” arXiv:2510.10504. Preprint, arXiv, October 12, 2025. https://doi.org/10.48550/arXiv.2510.10504 . ↩