

A Universal Property of the Gray Tensor - Campion

(March 16, Ea E T)

Introduction

This note follows a talk by Tim Campion on a *universal property of the gray tensor*^[1] at the FRG workshop on higher categories and geometry.

📖 Motivational Principle

Slogan: "Things get simpler at ∞ "

- **E.g.** Higher affineness

Analogy: "Things get simpler at $(\infty, 1)$ "

- **E.g.** An $(\infty, 1)$ -topos X is a locally presentable $(\infty, 1)$ -category such that $X_{/(-)} : X^{op} \rightarrow (\infty, 1)\text{Cat}$ preserves limits.

📖 Problem

1. Things seem complicated!
2. Things don't seem simpler

1. **E.g.** The description $(\infty, \infty)\text{Cat} = \lim_n (\infty, n)\text{Cat} = \text{colim}_n^{\text{Pr}^L} (\infty, n)\text{Cat}$ doesn't make the world of (∞, ∞) -categories simpler than just a pile of (∞, n) -categories, for $n \geq 0$.

📖 Solution?

How do (∞, m) -categories and (∞, n) -categories interact?

📖 Examples of Relations between $(\infty, m)\text{Cat}$ and $(\infty, n)\text{Cat}$

- **(1)** We have the inclusion $(\infty, n)\text{Cat} \hookrightarrow (\infty, n+1)\text{Cat}$
- **(2)** We have the suspension $\Sigma : (\infty, n)\text{Cat} \rightarrow (\infty, n+1)\text{Cat}$
- **(3)** We have the Gray tensor product $(\infty, m)\text{Cat} \times (\infty, n)\text{Cat} \xrightarrow{\boxtimes} (\infty, m+n)\text{Cat}$
- **(4)** We have the join $(\infty, m)\text{Cat} \times (\infty, n)\text{Cat} \xrightarrow{\star} (\infty, m+1+n)\text{Cat}$

Key Idea: We want to organize the data of (∞, ∞) -categories using the *gray tensor product* and *join*.

📖 Different Gray Tensors

- **Complcial (Verity):** The gray tensor product of complicial sets A and B , denoted $A \boxtimes B$, has underlying simplicial set $A \times B$, with a complicated combinatorial description of the markings.
- **Comical (Campion, Doterty, Kapulkin, Maehara^[2]):** Certain marked cubical sets, where for A, B cubical sets, $A \otimes B$ as an underlying cubical set is given naturally in terms of the cubical structure, and a cell $a \otimes b$ is marked if a or b is.

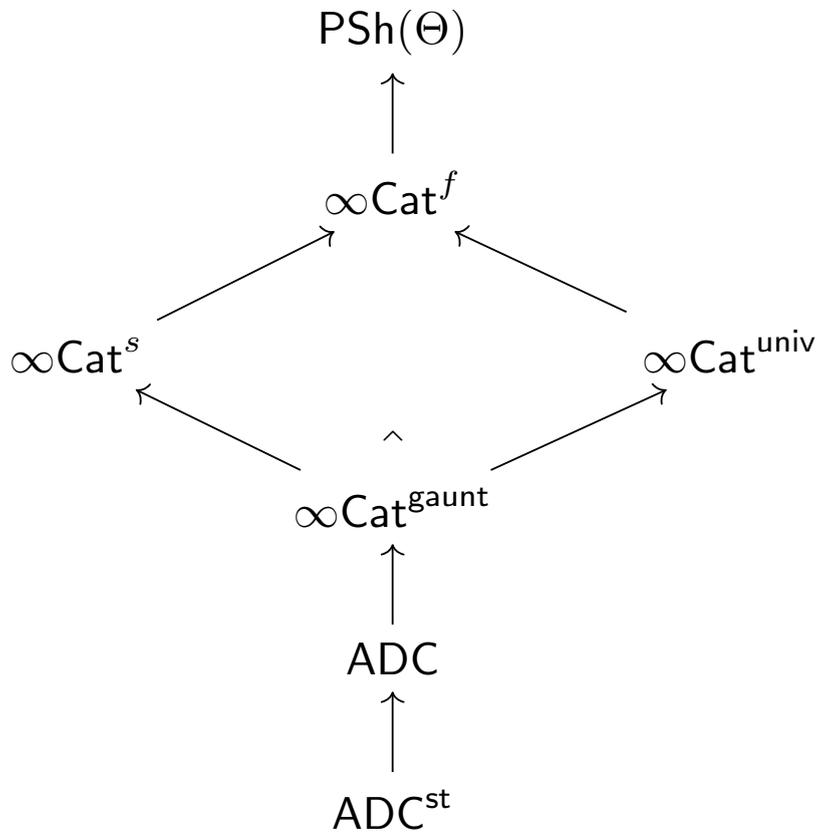
📖 Approach (Maehara, Campion)

Import the Gray tensor product \boxtimes from (strict) ∞ -categories, with description in terms of augmented directed complexes due to Steiner (c.f. [Strict Inf-Cats](#)). Specifically, one can show that $(\infty, \infty)\text{Cat}$ is a localization of space-valued pre-sheaves on (the (strict) 1-category of) augmented directed complexes (really the *strong Steiner complexes*), and then import the gray tensor product from the natural description of the gray tensor product on augmented directed complexes.

📖 Gaunt (∞, ∞) -category

An (∞, ∞) -category \mathcal{C} is **gaunt** if the space $\text{Map}(D^n, \mathcal{C})$ is 0-truncated for all n .

Intuition: Gaunt (∞, ∞) -categories are those with no non-trivial equivalences. In particular, all gaunt (∞, ∞) -categories are **strict** ∞ -categories. We have a pullback diagram



so that the *Gaunt* (∞, ∞) -categories are precisely the *strict univalent* (∞, ∞) -categories. Here $(\infty, \infty)\text{Cat} =: \infty\text{Cat}^{\text{univ}}$ are the univalent (∞, ∞) -categories. The Strong Steiner complexes admit a natural description of the Gray tensor product, which through Day convolution induces a monoidal structure on pre-sheaves on Strong Steiner complex.

- **Key Point:** This result is an instance of the **Day reflection theorem**.

Question

What needs to be shown for this to descend along the localizations?

- **Answer:** If L is a localization functor from a monoidal $(\infty, 1)$ -category, we need that if $f : A \rightarrow B$ is a map such that Lf is an equivalence, then for any object C , $L(C \otimes f)$ and $L(f \otimes C)$ are also equivalences.

Question

But what is \boxtimes *reallllllllly*?

Conjecture

Let $(\mathcal{A}, \otimes, \mathbb{1}) \in \text{Alg}_{\mathbb{E}_1}(\text{Pr})$. Then we have a pullback diagram

$$\begin{array}{ccc}
 \text{Fun}^{L, \mathbb{E}_1} \left(((\infty, \infty)\text{Cat}, \boxtimes), (\mathcal{A}, \otimes) \right) & \xrightarrow{\quad} & \text{Fun}^{L, \mathbb{E}_0} \left((\infty, 2)\text{Cat}, \mathcal{A} \right) \\
 \downarrow & \lrcorner & \downarrow \\
 \text{Fun}^{L, \mathbb{E}_0} \left((\infty, 1)\text{Cat}, \mathcal{A} \right) & \xrightarrow{\quad} & \text{Fun}^{L, \mathbb{E}_0} \left((\infty, 1)\text{Cat} \otimes (\infty, 1)\text{Cat}, \mathcal{A} \right)
 \end{array}$$

where $(\infty, 1)\text{Cat} \otimes (\infty, 1)\text{Cat} \simeq \infty\text{DbfCat}$, and $\infty\text{DbfCat} \rightarrow (\infty, 2)\text{Cat}$ is *left adjoint to the squares functor*. Here the bottom map sends $\Gamma : (\infty, 1)\text{Cat} \rightarrow \mathcal{A}$ to the composite

$$(\infty, 1)\text{Cat} \otimes (\infty, 1)\text{Cat} \xrightarrow{\Gamma \otimes \Gamma} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\otimes} \mathcal{A}$$

using the monoidal structure on \mathcal{A} .

- **Note:** A functor being \mathbb{E}_1 means it is *strict monoidal*, while being \mathbb{E}_0 means it preserves the unit.

Intuition: we need only specify the data of a cocontinuous functor $(\infty, 1)\text{Cat} \rightarrow \mathcal{A}$ functor such that the composite

$$\infty\text{DbfCat} \simeq (\infty, 1)\text{Cat} \otimes (\infty, 1)\text{Cat} \xrightarrow{\Gamma \otimes \Gamma} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\otimes} \mathcal{A}$$

factors through the left adjoint $\infty\text{DbfCat} \rightarrow (\infty, 2)\text{Cat}$ to the squares functor $\text{Sq} : (\infty, 2)\text{Cat} \rightarrow \infty\text{DbfCat}$.

- **Key Point:** The left adjoint to Sq is the restriction of the Gray tensor product $\boxtimes : (\infty, 1)\text{Cat} \otimes (\infty, 1)\text{Cat} \rightarrow (\infty, 2)\text{Cat}$.

🔍 Observations on the Conjecture

- An oriented category (in the sense of Gepner-Heine) is a module over $((\infty, \infty)\text{Cat}, \boxtimes)$, which can be realized as a left adjoint functor

$$((\infty, \infty)\text{Cat}, \boxtimes) \rightarrow \text{End}(\mathcal{M})$$

The conjecture says that this consists of the data of

- **(i)** a left adjoint functor $\Gamma : (\infty, 1)\text{Cat} \rightarrow \text{End}(\mathcal{M})$, which is determined by the actions on Δ , namely by a co-Segal object $\Gamma|_{\Delta} : \Delta \rightarrow \text{End}(\mathcal{M})$
- **(ii)** a left adjoint functor $\Gamma \otimes \Gamma : \infty\text{DbfCat} \rightarrow \text{End}(\mathcal{M})$, determined by a bi-cosimplicial Segal object $\Gamma \otimes \Gamma|_{\Delta \times \Delta} : \Delta \times \Delta \rightarrow \text{End}(\mathcal{M})$ which is dual to an object arising from the squares functor construction.

Recall: The squares functor is induced by a functor $\Delta \times \Delta \rightarrow \Theta_2$ sending $([n], [m])$ to the Gray tensor product $[n] \boxtimes [m]$.

🔍 Question

What is the analogous universal property for the gray join?

References

1. Campion, Timothy. "The Gray Tensor Product of (∞, n) -Categories." arXiv:2311.00205. Preprint, arXiv, November 1, 2023. <https://doi.org/10.48550/arXiv.2311.00205>. ↩
2. Campion, Tim, Chris Kapulkin, and Yuki Maehara. "A Cubical Model for (∞, n) -Categories." *Geometry & Topology* 29, no. 3 (2025): 1115–70. <https://doi.org/10.2140/gt.2025.29.1115>. ↩