Math 515-2: sheaves
Problem set 01: posets

Definition. A 0-category is a category enriched in $\{\emptyset, *\}$. (These are also called preorders.)

1. Every 0-category is equivalent to a poset.

2. A poset has colimits if and only if it has arbitrary joins (least upper bounds). It has limits if and only if it has arbitrary meets (greatest lower bounds).

3. A poset has colimits if and only if it has limits.

4. A presentable poset is small.

5. A locale is a presentable poset such that joins distribute over finite meets:

$$x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} x \wedge y_i.$$

If X is a topological space, then $\mathcal{O}\mathrm{p}(X)$ is a locale. (These are also called frames sometimes, but since we haven't defined the morphisms between these objects, it doesn't matter.)

6. Give a presentable poset which is not a locale.

7. Define a κ-compact object of a poset. Define what it means for a poset to be κ-compactly generated.

8. Show that any locale is a localization of a compactly generated locale where, moreover, the localization functor is left exact (preserves finite joins).

9. Let $F\colon P \to Q$ be a functor of posets. Assume that P and Q have joins and that F preserves them. If P is presentable, then F admits a right adjoint.

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