Math 515-1: Derived commutative rings
Problem set 01

1. Show that if $A_\bullet$ is a simplicial group, then it is a Kan complex.

2. Let X be a topological space. Define the singular simplicial set $\mathrm{Sing}_\bullet(X)$. Show that it is a Kan complex.

3. We can consider the ordered set $[n] = \{0 < 1 < \cdots < n\}$ as a category. Taken together, these form a cosimplicial category, i.e., a functor $\Delta \to \mathcal{C}\mathrm{at}$. Given a category $\mathcal{C}$, we can consider the assignment $[n] \mapsto \mathrm{Hom}_{\mathcal{C}\mathrm{at}}([n], \mathcal{C})$, which forms a simplicial set, called the nerve of $\mathcal{C}$ and denoted by $\mathrm{N}_\bullet(\mathcal{C})$. Show that $\mathrm{N}_\bullet(\mathcal{C})$ is a weak Kan complex.

4. Find necessary and sufficient conditions for a weak Kan complex to be isomorphic to $\mathrm{N}_\bullet(\mathcal{C})$ for a category $\mathcal{C}$.

5. Let $\mathcal{C}$ be a category. Find necessary and sufficient conditions for $\mathrm{N}_\bullet(\mathcal{C})$ to be a weak Kan complex.

6. Let M be a monoid. We define $\mathrm{B}M$ as the category with one object $*$ and where $\mathrm{Hom}_{\mathrm{B}M}(*, *) = M$, with composition given by multiplication in the group. Let $\mathrm{B}_\bullet M$ be the nerve of the category $\mathrm{B}M$. Describe the simplices, face maps, and degeneracies of $\mathrm{B}M$ in terms of the elements of M.

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