

Category, space, type - Benjamin Antieau

17. Urysohn's lemma

Definition 18.1 (Normal topological space). Recall that a topological space X is normal if for every pair $A, B \subseteq X$ of closed subsets such that $A \cap B = \emptyset$, there exist open subsets U, V such that

- (i) $A \subseteq U$,
- (ii) $B \subseteq V$, and
- (iii) $U \cap V = \emptyset$.

Example 18.2. We proved in Example 8.19 that compact Hausdorff spaces are normal.

The following “lemma” and Tychonoff's theorem form the main core of major difficult results at the heart of point-set topology.

Theorem 18.3 (Urysohn's lemma). *If X is a normal topological space and A, B are disjoint closed subsets, then there is a continuous function $f: X \rightarrow \mathbf{R}$ such that $f(A) = 0$ and $f(B) = 1$.*

Proof. We will use a dyadic approach. Let $D = \mathbf{Z}[\frac{1}{2}]$ be the set of dyadic numbers, meaning those rational numbers which can be represented as a fraction $\frac{m}{2^n}$ for $m, n \geq 0$.

We first define a function $F: D \rightarrow \mathcal{O}(X)$. Thus, for a dyadic s , we will have an open $F(s) \subseteq X$. In addition if $s < t$ are dyadic numbers, we will have $F(s) \subseteq F(t)$. We will use F in a moment to construct the function f .

If $s > 1$, let $F(s) = X$. If $s = 1$, we let $F(s) = X \setminus B$. If $s < 0$, we let $F(s) = \emptyset$. If $s = 0$, we pick an open subset U containing A and such that $\overline{U} \cap B = \emptyset$. We can do this by normality: choose opens U and V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. Then, U is contained in the closed subset $X \setminus V$ and so its closure is too. But, $X \setminus V \subseteq X \setminus B$, so that \overline{U} does not intersect B . Now, set $F(0) = U$.

We will argue by induction on the exponent of the denominator. Suppose that $n \geq 1$ and that F has been defined on all dyadic numbers of the form $\frac{2m+1}{2^\ell}$ for $0 \leq \ell < n$. By induction $\overline{F(\frac{2m}{2^n})} \subseteq F(\frac{2m+2}{2^n})$.¹ Choose $F(\frac{2m+1}{2^n})$ such that it contains $\overline{F(\frac{2m}{2^n})}$ and such

¹Note that these are not in lowest terms.

that $\overline{F(\frac{2m+1}{2^n})} \subseteq F(\frac{2m}{2^n})$. We can do this by normality applied to the pair of closed subsets $\overline{F(\frac{2m}{2^n})}$ and $X \setminus F(\frac{2m+2}{2^n})$ as in the preceding paragraph.

Now, we define $f: X \rightarrow \mathbf{R}$ by letting

$$f(x) = \inf\{s \in D : x \in F(s)\}.$$

We have $A \subseteq F(s)$ for all $s \in D$. Thus, $f(x) = 0$ for $x \in A$ since we are taking the infimum of D itself. If $x \in B$, then $x \in F(s)$ for $s > 1$ and then $s \notin F(1)$. Thus, we are taking the infimum of the dyadic numbers which are more than 1, so if $x \in B$, then $f(x) = 1$. We have constructed a function with the appropriate values on A and B . It remains to show that it is continuous.

For this, we note that the open sets (s, ∞) and $(-\infty, t)$ for $s, t \in D$ form a subbasis for the euclidean topology on \mathbf{R} . It is thus enough to prove that the sets of the form $f^{-1}((s, \infty))$ and $f^{-1}((-\infty, t))$ are open in X for $s, t \in D$. In fact, it is enough to show that $f^{-1}((-\infty, s))$ is open and $f^{-1}([-\infty, s])$ is closed, and that is what we will do.

We have that $f^{-1}((-\infty, s)) = \{x \in X : f(x) < s\}$ by definition. If $f(x) < s$, then there exists t such that $f(x) < t < s$ and hence $x \in F(t)$. In other words,

$$\{x \in X : f(x) < s\} = \bigcup_{t < s} F(t),$$

which is a union of open sets and is thus open.

Similarly, $f^{-1}((-\infty, s]) = \{x \in X : f(x) \leq s\}$. The reader can check that this set is

$$\bigcap_{t > s} F(t).$$

This is an intersection of open sets, which is not yet that helpful. But, we claim that in fact

$$\bigcap_{t > s} F(t) = \bigcap_{t > s} \overline{F(t)},$$

which will show that this set is closed and complete the proof. Of course, $F(t) \subseteq \overline{F(t)}$, so the left-hand side is contained in the right-hand side. But, by construction, if $t > s$, then there is a dyadic r such that $t > r > s$ and $F(r) \subseteq \overline{F(r)} \subseteq F(t)$. By a ‘‘squeeze’’ argument, the result follows.

Specifically, we can replace the intersections above by

$$\bigcap_{n \geq 1} F(s + \frac{1}{2^n}) \quad \text{and} \quad \bigcap_{n \geq 1} \overline{F(s + \frac{1}{2^n})}.$$

We have $\overline{F(s + \frac{1}{2^{n+1}})} \subseteq F(s + \frac{1}{2^n})$. Thus,

$$\bigcap_{n \geq 1} F(s + \frac{1}{2^n}) \subseteq \bigcap_{n \geq 1} \overline{F(s + \frac{1}{2^n})} = \bigcap_{n \geq 1} \overline{F(s + \frac{1}{2^{n+1}})} \subseteq \bigcap_{n \geq 1} F(s + \frac{1}{2^n}).$$

The middle equality used that in the intersection we can just omit the first term corresponding to $\overline{F(s + \frac{1}{2})}$ since the sets are nested. \square

So, where to go from here? There are lots of consequences of Urysohn's lemma. Most concrete is the metrization theorem.

Theorem 18.4 (Urysohn's metrization theorem). *Suppose that X is a regular T_1 space and suppose that it has a countable basis. Then, X is metrizable.*

One uses Urysohn's lemma to construct enough functions like $d(x, -)$ on X .

Here is another direction where the philosophy of Urysohn's lemma is used. It is another duality theorem. It says that compact Hausdorff spaces X can be recovered from $C(X, \mathbf{R}) = \text{Hom}_{\text{Top}}(X, \mathbf{R})$. Of course, for this to be true, there need to be enough functions to distinguish between points. Urysohn's lemma provides this.

Definition 18.5. A commutative ring is a set R with addition $+$ and multiplication \cdot satisfying the same axioms as \mathbf{Z} satisfies. Addition is commutative and associative; there is an additive unit written 0 ; every element x has an additive inverse $-x$. Multiplication is commutative and associative; there is a multiplicative unit 1 . We also require that $x(y + z) = xy + xz$.

Definition 18.6. A field is a commutative ring k such that $1 \neq 0$ and additionally if $x \neq 0$ then x has a multiplicative inverse x^{-1} .

There is an evident notion of maps of commutative rings. A commutative \mathbf{R} -algebra is a commutative ring A with a fixed map $u_A: \mathbf{R} \rightarrow A$. We let $\text{CAlg}_{\mathbf{R}}$ denote the category of commutative \mathbf{R} -algebras. The morphisms are commutative ring maps $f: A \rightarrow B$ such that $u_A = u_B \circ f$.

Definition 18.7. Since we can add and multiply continuous \mathbf{R} -valued functions, $C(X; \mathbf{R})$ is a commutative ring. Since we can multiply these by scalars, it has the structure of an \mathbf{R} -algebra. If $f: X \rightarrow Y$ is continuous, then composition with f defines a commutative \mathbf{R} -algebra map $C(Y; \mathbf{R}) \rightarrow C(X; \mathbf{R})$. All in all, we have defined a functor

$$C(-; \mathbf{R}): \mathbf{Top}^{\text{op}} \rightarrow \mathbf{CAlg}_{\mathbf{R}}.$$

Definition 18.8 (Ideals). Let A be a commutative ring. An ideal in A is an additive subgroup $I \subseteq A$ such that if $x \in I$ and $y \in A$ the product $xy \in I$. If I is an ideal, then the quotient abelian group A/I is a commutative ring too and $A \rightarrow A/I$ is a commutative ring map.

Definition 18.9 (Maximal ideals). An ideal I in a commutative ring is called maximal if A/I is a field. Equivalently, $I \neq A$ and the only ideals containing I are I and A .

Example 18.10. If k is a field and $f: A \rightarrow k$ is a *surjective* ring homomorphism, then $\ker(f)$ is a maximal ideal.

Example 18.11. If p is a prime number, then the ideal $(p) = \{n \in \mathbf{Z} : p|n\}$ is a maximal ideal. The quotient ring is the field \mathbf{F}_p .

Example 18.12. Suppose that X is a topological space and $x \in X$ is a point. Then, the function $\text{ev}_x: C(X; \mathbf{R}) \rightarrow \mathbf{R}$ which satisfies $\text{ev}_x(f) = f(x)$ is a surjective ring homomorphism. The kernel is the maximal ideal of functions vanishing at x .

Definition 18.13. Let A be a commutative \mathbf{R} -algebra. Let $\mathbf{RSpec}(A)$ be the set of maximal ideals in A arising from commutative \mathbf{R} -algebra maps $A \rightarrow \mathbf{R}$. (Note that any such ring map is surjective since it sends 1 to 1 and we can scale.) If $f: A \rightarrow B$ is a commutative \mathbf{R} -algebra map, then there is an induced function $\mathbf{RSpec}(f): \mathbf{RSpec}(B) \rightarrow \mathbf{RSpec}(A)$. Now, we can give $\mathbf{RSpec}(A)$ a topology as follows. If $x \in A$, we let $\mathbf{RSpec}(A)_x$ be the set of maximal ideals *not* containing x . (Equivalently, it corresponds to \mathbf{R} -algebra maps $g: A \rightarrow \mathbf{R}$ such that $g(x) \neq 0$.) We have $\mathbf{RSpec}(A)_{xy} = \mathbf{RSpec}(A)_x \cap \mathbf{RSpec}(A)_y$. We also have $\mathbf{RSpec}(A)_x = \mathbf{RSpec}(A)$. Thus, the sets of this form form a basis for a topology on $\mathbf{RSpec}(A)$. With respect to these topologies, $\mathbf{RSpec}(f)$ is continuous. In other words, we have defined a functor

$$\mathbf{RSpec}: \mathbf{CAlg}_{\mathbf{R}} \rightarrow \mathbf{Top}^{\text{op}}.$$

Exercise 18.14. Prove that $\mathbf{R}\text{Spec}(A)$ is T_1 for any commutative \mathbf{R} -algebra A .

In general, neither functor $C(-; \mathbf{R})$ nor $\mathbf{R}\text{Spec}(-)$ is that helpful. However, if X is compact and Hausdorff, then it turns out that $\mathbf{R}\text{Spec}(C(X; \mathbf{R}))$ is homeomorphic to X . More generally, we have the following theorem.

Theorem 18.15 (Gelfand duality). *Restricted to CHaus^{op} , the functor $C(-; \mathbf{R})$ is fully faithful. Its essential image consists of precisely the real unital commutative C^* -algebras (certain complete normed \mathbf{R} -vector spaces with multiplication).*

Corollary 18.16. *Let X and Y be compact Hausdorff spaces. Then, $\text{Hom}_{\text{Top}}(X, Y) \cong \text{Hom}_{\text{CAlg}_{\mathbf{R}}}(C(Y; \mathbf{R}), C(X; \mathbf{R}))$.*

Thus, a large part of topology could be developed using opens, but only continuous functions. The cost is that we would not be able to access non-Hausdorff spaces.

Finally, let me remark on one more construction. We can replace \mathbf{R} with \mathbf{F}_2 with the discrete topology. If X is a topological space, then $C(X; \mathbf{F}_2)$ is a commutative \mathbf{F}_2 -algebra. If A is a commutative \mathbf{F}_2 -algebra, then write $\mathbf{F}_2\text{Spec}(A)$ for the set of kernels of commutative \mathbf{F}_2 -algebra maps $A \rightarrow \mathbf{F}_2$.

Definition 18.17 (Boolean). Say that a commutative \mathbf{F}_2 -algebra A is Boolean if $x^2 = x$ for all $x \in A$. Let $\text{CAlg}_{\mathbf{F}_2}^{\varphi=1}$ be the category of Boolean commutative \mathbf{F}_2 -algebras. Note that $C(X; \mathbf{F}_2)$ is Boolean.

Definition 18.18. Say that a compact Hausdorff topological space X is totally disconnected if the only connected subsets are singletons. Write Stone for the category of totally disconnected compact Hausdorff spaces; these are called Stone spaces (or profinite spaces).

Exercise 18.19. Show that the Cantor set is totally disconnected, that the one-point compactification of \mathbf{N} is totally disconnected, but that $[0, 1]$ is not.

Theorem 18.20 (Stone duality). *The functors $C(-; \mathbf{F}_2)$ and $\mathbf{F}_2\text{Spec}(-)$ restrict to inverse equivalences*

$$C(-; \mathbf{F}_2): \text{Stone}^{\text{op}} \rightleftarrows \text{CAlg}_{\mathbf{F}_2}^{\varphi=1} : \mathbf{F}_2\text{Spec}.$$

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