Category, space, type - Benjamin Antieau
17. Tychonoff's theorem

`Definition 17.1.` Let $\{X_s\}_{s\in S}$ be a family of sets indexed by a set S. The product of the family is

$$\prod_{s\in S} X_s,$$

often abbreviated as $\prod_s X_s$, and consists of all S-indexed tuples $(x_s)_{s\in S}$ where $x_s \in X_s$.

`Example 17.2.` If $S = \emptyset$, then the product $\prod_{s\in S} X_s$ consists of exactly one point, corresponding to the unique, empty tuple.

`Example 17.3.` If S is finite, say $S = \{1, \ldots, n\}$, then

$$\prod_{s\in S} X_s = \prod_{i=1}^{n} X_i.$$

This is the usual product.

`Remark 17.4.` If S is infinite, then this construction is likely new. However, some special cases are probably familiar. For example, suppose that we consider $S = \mathbf{N}$ and $X_n = \{0, 1\}$. Then, the product

$$\prod_{n\in\mathbf{N}} X_n = \prod_{n\geqslant 0}\{0, 1\}$$

is the set of countable binary sequences $(b_0, b_1, b_2, \ldots)$. In this case, the product is the set of functions $\mathbf{N} \to \{0, 1\}$. Even though each $\{0, 1\}$ is very large, this set is uncountable.

`Construction 17.5 (General product topology).` Suppose now that $\{X_s\}_{s\in S}$ is a family of *topological spaces* indexed by a set S. What topology shall we impose on $X = \prod X_s$? We are guided by the principal of understanding the appropriate universal property. As a set, X has the following universal property. A function $W \to X$ is "the same" as a family of functions $(f\colon X \to X_s)_{s\in S}$. Specifically, X comes with a family of projection maps $p_s\colon X \to X_s$ and composition with these induces a bijection

$$\mathrm{Hom}_{\mathbf{Set}}(W, X) \xrightarrow{f\mapsto(p_s\circ f)} \prod_{s\in S}\mathrm{Hom}_{\mathbf{Set}}(W, X_s).$$

We aim for the same universal property for the (infinite) product topology.

This means we at the very least require that the production maps $X \to X_s$ be continuous. Thus, if $U \subseteq X_s$ is open, then $p_s^{-1}(U) \subseteq X$ should be open. We let $\mathcal{B}$ be the set of all subsets of the form $p_s^{-1}(U)$ where s ranges over the elements of S. We let $\mathcal{U} \subseteq \mathbf{P}(X)$ be the topology generated by the subbasis $\mathcal{B}$ and declare this to be the product topology.

`Exercise 17.6.` Prove that $(X, \mathcal{U})$ and the continuous projection maps $p_s \colon X \to X_s$ has the desired universal property: that to give a continuous function $f \colon W \to X$ is "the same" as giving continuous functions $f_s \colon W \to X_s$, one for each $s \in S$.

`Example 17.7.` Suppose that S is infinite and that $U_s \subseteq X_s$ is an open set for each $s \in S$. It is typically **not** the case that

$$\prod U_s \subseteq X_s$$

is open in the product topology. First of all, there is no reason for it to be as it is an infinite intersection

$$\bigcap p_s^{-1}(U_s)$$

of open subsets. For a specific example, consider the topological space $X = \prod_{\mathbf{N}} \{0, 1\}$ of countable binary sequences. We equip each $\{0, 1\}$ with the discrete topology. In particular, $\{0\} \subseteq \{0, 1\}$ is open. But, this subset cannot be a union of finite intersections of open sets of the form $p_n^{-1}(V_n)$. Indeed, the latter contains all "tails". Specifically, each $p_n^{-1}(V_n)$ places no restrictions at all on the binary digits in places greater than n. We we take only finitely many intersections, then we see that there are not restrictions on the binary digits in large enough degree. In particular, we cannot cut down to just tails of the form $(0, 0, 0, \ldots)$.

`Remark 17.8.` In particular, the product topology on the set of countable binary sequences is not the discrete topology!

`Remark 17.9.` The topology generated by the basis of subsets of the form $\prod U_s \subseteq \prod X_s$ where each $U_s \subseteq X_s$ is open is called the box topology.

Here is another way to see this; it is one of the most heavily-used theorems from point-set topology.

`Theorem 17.10 (Tychonoff).` *If $\{X_s\}_{s\in S}$ is a family of topological spaces indexed by a set S and if each X_s is compact, then*

$$X = \prod_{s\in S} X_s$$

is compact.

To prove it, we need a little terminology.

`Definition 17.11.` Let Y be a set. Let $\mathcal{F} \subseteq \mathbf{P}(Y)$ be a set of subsets of Y. We say that the family $\mathcal{F}$ has finite character if the following two conditions hold:

(i) if $A \in \mathcal{F}$ is an element of the family $\mathcal{F}$ and if $B \subseteq A$ is finite, then $B \in \mathcal{F}$;

(ii) if $A \subseteq Y$ (i.e., $A \in \mathbf{P}(Y)$) and if every finite subset $B \subseteq A$ is in $\mathcal{F}$, then $A \in \mathcal{F}$.

Tukey's lemma is that each non-empty family $\mathcal{F}$ of finite character has a maximum element. It is equivalent to Zorn's lemma.

Proof. I learned this from Kelley's book *General Topology*, where it is attributed to Bourbaki.

First, some terminology. Say a collection $\mathcal{C}$ of subsets of a set X has the finite intersection property if for all finitely many subsets $Y_1, \dots, Y_n \in \mathcal{C}$ the intersection $Y_1 \cap \cdots \cap Y_n$ is nonempty.

If X is a topological space, then X is compact if and only if each family $\mathcal{K}$ of closed subsets of X which has the finite intersection property actually has a nonempty intersection:

$$\cap_{K\in\mathcal{K}} K \neq \emptyset.$$

This follows from De Morgan's laws.

Now, let $\mathcal{C}$ be any family of subsets of X with the finite intersection property. We will show that

$$\bigcap_{C\in\mathcal{C}} \overline{C}$$

is nonempty. The sets do not have to be open or closed or anything. But, this is enough to prove the theorem by the previous paragraph.

The class $\mathcal{F}$ of families of subsets of X with the finite intersection property has finite character. Note that $\mathcal{F} \subseteq \mathbf{P}(\mathbf{P}(X))$. In particular, by Tukey's lemma, there is a maximum element of $\mathcal{F}$, meaning a maximum family of subsets of X with the finite intersection property. So, we might as well assume that $\mathcal{C}$ is this maximum; indeed showing a *larger* intersection is nonempty is a *stronger* statement.

Suppose that $C \subseteq D \subseteq X$ is such that $C \in \mathcal{C}$. Then, any finite intersection involving D and elements of $\mathcal{C}$ must also be nonempty, so $D \in \mathcal{C}$. (Thus, $\mathcal{C}$ is upwards closed in $\mathbf{P}(X)$.) Also, if $C_0, C_1 \in \mathcal{C}$, then $C_0 \cap C_1 \in \mathcal{C}$. Also, if $D \subseteq X$ is such that $C \cap D \neq \emptyset$ for each $C \in \mathcal{C}$, then $D \in \mathcal{C}$.

For each $s \in S$, consider $p_s(\mathcal{C}) \subseteq \mathbf{P}(X_s)$. This has the finite intersection property since $\mathcal{C}$ does. There is in particular a point

$$x_s \in \bigcap_{C \in \mathcal{C}} \overline{p_s(C)}$$

by compactness of X_s. Pick one for each $s \in S$ and let x be the point whose s-coordinate is x_s for each $s \in S$.

By definition of closures, if $U \subseteq X_s$ is an open containing x_s, then $U \cap p_s(C)$ is nonempty for all $C \in \mathcal{C}$. It follows that $p_s^{-1}(U) \cap C$ is nonempty for each $C \in \mathcal{C}$. It follows from the previous paragraph that $p_s^{-1}(U) \in \mathcal{C}$. This is then true for intersections of any finite collection of sets of this form (for varying s).

If $V \subseteq X$ is an open and contains x, then it must intersect every $C \in \mathcal{C}$ since this is true for finite intersections of subbasis elements containing x. Thus, $x \in \cap_{C \in \mathcal{C}} \overline{C}$, as desired. □

`Remark 17.12.` In particular, we see that $\{0,1\}^{\mathbf{N}} = \prod_{\mathbf{N}} \{0,1\}$ is compact. The discrete topology on this uncountable set would be highly non-compact.

`Remark 17.13.` In fact, $\{0,1\}^{\mathbf{N}}$ is homeomorphic to the Cantor set. As a closed, bounded subset of $\mathbf{R}$, we know it must be compact.

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