

Proposition 16.1. *Suppose that X is a topological space. Then, X is sober if and only if the unit map $\eta_X: X \rightarrow |\mathbf{O}(X)|$ is a homeomorphism.*

Proof. Suppose that X is sober so that every irreducible closed set K has a unique generic point. Note that the irreducible closed subsets of X are in one-to-one correspondence with the indecomposables of $|\mathbf{O}(X)|$. Indeed, a closed subset $K \subseteq X$ is irreducible if and only if the open $X \setminus K$ is indecomposable in $\mathbf{O}(X)$. The function $X \rightarrow |\mathbf{O}(X)|$ can be interpreted as sending $x \in X$ to $X \setminus \overline{\{x\}}$, the complement of the closure of $\{x\}$. If each irreducible admits a generic point, then this assignment is surjective. If generic points are unique, then it is injective. Thus, if X is sober, then η_X is a continuous bijection. If $U \subseteq X$ is open, we have to show that $\eta_X(U)$ is open. By definition, $\eta_X(U)$ consists of the set of indecomposables $X \setminus \overline{\{x\}}$ as x ranges over the elements of U . This is precisely $|\mathbf{O}(X)|_U$. It follows that $\eta_X(U)$ is open, so that η_X is a homeomorphism. Conversely, if η_X is a homeomorphism, then every irreducible closed subset has a unique generic point, just from the bijectivity of η_X . \square

Definition 16.2. Say that a locale L is spatial if $\epsilon_L: \mathbf{O}(|L|) \rightarrow L$ is an isomorphism of locales.

Remark 16.3. The opens of $|L|$ are all of the form $|L|_s$ for $s \in L$. Thus, $\epsilon_L^*: L \rightarrow \mathbf{O}(|L|)$ is surjective. The spatiality condition is that the map is injective too. This means that if $s \neq t \in L$, then there is some indecomposable x such that either $s \not\leq x$ and $t \leq x$ or $s \leq x$ and $t \not\leq x$.

Lemma 16.4. *If X is a topological space, then $\mathbf{O}(X)$ is spatial.*

Proof. Indeed, suppose that $U \neq V$ as points in $\mathbf{O}(X)$. Then, since $\eta_X^{-1}(|\mathbf{O}(X)|_U) = U$ and $\eta_X^{-1}(|\mathbf{O}(X)|_V) = V$, so there is some indecomposable that witness that they are not the same. \square

Lemma 16.5. *Suppose that L is a locale. Then, $|L|$ is a sober topological space.*

Proof. Let $x \neq y \in |L|$. To prove that generic points are unique, it suffices to show that $\overline{\{x\}} \neq \overline{\{y\}}$. Since L is particular a lattice, we have $x \not\leq y$ or $y \not\leq x$. Assume that $x \not\leq y$. It follows that $y \in |L|_x$. But, $x \notin |L|_x$. Hence, y is not in the closure of $\{x\}$.

Now, suppose that $K \subseteq |L|$ is an irreducible closed subset. Consider the subset $S = \{s \in L : K = |L| \setminus |L|_s\}$. The set S is nonempty since every open subset of $|L|$ occurs as some $|L|_s$. Let $t = \vee S$. Note that $t \in S$ since $|L|_t = |L|_{\vee S} = \cup_{s \in S} |L|_s$, but each $|L|_s$ is $|L| \setminus K$. Thus, t is a maximum for S . We have that $t \neq \top$ since in that case $|L|_{\top} = |L|$ and $K = \emptyset$, which contradicts irreducibility. If $t = x \wedge y$ for $x, y \in |L|$, then $|L|_t = |L|_x \cup |L|_y$. Let $K_x = |L| \setminus |L|_x$ and $K_y = |L| \setminus |L|_y$. We must have $K = K_x \cup K_y$. By irreducibility, either $K_x = K$ or $K_y = K$. But, this means x or y is in S . By maximality of t , we must find $t = x$ or $t = y$. \square

Exercise 16.6. Prove that the functors $\mathbf{O}: \mathbf{Top} \rightarrow \mathbf{Loc}$ and $\mathbf{Top} \leftarrow \mathbf{Loc}: |-|$ are adjoint via the unit and counit natural transformations constructed previously.

Definition 16.7. Consider an adjunction $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$. If G is fully faithful, we say that the adjunction is a localization. If F is fully faithful, we say the adjunction is a colocalization.

Remark 16.8. Terminology differs. Some might say that F is a localization. Others might say that the composition $G \circ F: \mathcal{C} \rightarrow \mathcal{C}$ is a localization functor.

Exercise 16.9. Show that an adjunction $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ with unit η and counit ϵ is a localization if and only if the counit map $\epsilon_Y: F(G(Y)) \rightarrow Y$ is an isomorphism for every $Y \in \mathcal{D}$. Show that it is a colocalization if and only if the unit map $\eta_X: X \rightarrow G(F(X))$ is an isomorphism for every $X \in \mathcal{C}$.

Construction 16.10. The functor $\mathbf{O}: \mathbf{Top} \rightarrow \mathbf{Loc}$ takes values in $\mathbf{Loc}^{\text{spatial}}$. The functor $\mathbf{Top} \leftarrow \mathbf{Loc}: |-|$ takes values in $\mathbf{Top}^{\text{sober}}$. We can view $\mathbf{Top}^{\text{sober}}$ as a localization of \mathbf{Top} . Indeed, the fully faithful inclusion $\mathbf{Top} \leftarrow \mathbf{Top}^{\text{sober}}$ has a left adjoint (soberification) given by $X \mapsto |\mathbf{O}(X)|$. Similarly, we can view $\mathbf{Loc}^{\text{sober}}$ as a colocalization of \mathbf{Loc} as the fully faithful inclusion $\mathbf{Loc}^{\text{spatial}} \subseteq \mathbf{Loc}$ has a right adjoint (spatialization) given by $L \mapsto \mathbf{O}(|L|)$.

Theorem 16.11. *The functors \mathbf{O} and $|-|$ restrict to inverse equivalences $\mathbf{Top}^{\text{sober}} \simeq \mathbf{Loc}^{\text{spatial}}$.*

Proof. By the lemmas above, O and $|-|$ factor through $\mathbf{Loc}^{\text{spatial}}$ and $\mathbf{Top}^{\text{sober}}$, respectively. We abuse notation and consider the induced adjunction

$$O: \mathbf{Top}^{\text{sober}} \rightleftarrows \mathbf{Loc}^{\text{spatial}}: |-|.$$

We have seen that the unit and counit maps are equivalences. Thus, O and $|-|$ are inverse equivalences of categories, by definition.¹ \square

Example 16.12. If X is sober and $U \subseteq X$ is open, then U is sober. If X is the union of sober open subspaces, then X is sober.

Remark 16.13. There are non-spatial locales. One example is given by the locale of so-called regular open subsets of \mathbf{R} . An open $U \subseteq \mathbf{R}$ is regular open if U is the interior of its closure. For example, $(0, 1)$ is a regular open, but $(0, 1) \cup (1, 2)$ is not. The indecomposable opens $\mathbf{R} \setminus \{x\}$ are not regular. One can show that the regular opens form a local $O^{\text{reg}}(\mathbf{R})$ (although the join is not generally given by the union), but that $|O^{\text{reg}}(\mathbf{R})|$ is empty, so certainly the counit map $O(|O^{\text{reg}}(\mathbf{R})|) \rightarrow O^{\text{reg}}(\mathbf{R})$ is not an isomorphism.

Remark 16.14. Locales are the starting point of what is called topos theory and encompasses the theory of sheaves. In fact, another name for a locale is a 0-topos.

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¹Alternatively, O is fully faithful and essentially surjective.