Category, space, type - Benjamin Antieau
15. Locales II

Example 15.1. The real numbers. Let $\mathrm{O}(\mathbf{R})$ be the locale associated to the topological space $\mathbf{R}$ with its usual Euclidean topology. What objects of $\mathrm{O}(\mathbf{R})$ are indecomposable? Equivalently, which opens U in $\mathbf{R}$ have the property that if V and W are open subsets such that $U = V \cap W$, then $U = V$ or $U = W$? These are precisely the sets of the form $U_x = \mathbf{R} \setminus \{x\}$ for some $x \in \mathbf{R}$. This agrees with the function in Construction 14.18, so we see that $\mathbf{R} \to |\mathrm{O}(\mathbf{R})|$ is a continuous bijection. Given an open $V \subseteq \mathbf{R}$, given an element of $\mathrm{O}(\mathbf{R})$, the open set $|\mathrm{O}(\mathbf{R})|_V$ consists of the set of indecomposables U_x such that $V \nleq U_x$. But, $V \nleq U_x$ if and only if $x \in V$. Thus, under the identification $\mathbf{R} \cong |\mathrm{O}(\mathbf{R})|$ above, the open $|\mathrm{O}(\mathbf{R})|_V$ of $|\mathrm{O}(\mathbf{R})|$ corresponds to V itself. Hence, $\mathbf{R} \to |\mathrm{O}(\mathbf{R})|$ is a homeomorphism.

The example above suggests the following definition.

Definition 15.2 (Irreducibles). Let X be a topological space and let $K \subseteq X$ be a closed subset. Say that K is irreducible if it is nonempty and it cannot be written as the union of two proper closed subsets of X.

Definition 15.3 (Generic points). Let X be a topological space and let $K \subseteq X$ be a closed subset. Say $x \in K$ is a generic point of K if $\overline{\{x\}} = K$.

Definition 15.4 (Sober spaces). A topological space X is sober if every irreducible closed subset has a unique generic point.

Exercise 15.5. Show that if X is a topological space and $x \in X$, then the closure $\overline{\{x\}}$ is irreducible.

Exercise 15.6. Show that if X is a Hausdorff space, then it is sober.

Exercise 15.7. Characterize the sober finite topological spaces.

Example 15.8. Consider the cofinite topology on $\mathbf{N}$. This topological space is not sober. The irreducible closed subsets are either of the form $\{n\}$ for some $n \in \mathbf{N}$ or $\mathbf{N}$ itself. But, $\mathbf{N}$ has no generic point. In this case, sobriety fails because of a failure of existence of generic points. This space is T_1 but not T_2.

`Exercise 15.9.` Uniqueness of generic points can also fail. Find an example.

`Exercise 15.10.` Let X be a topological space. Show that the continuous function $X \to |\mathrm{O}(X)|$ is a homeomorphism if and only if X is sober.

`Remark 15.11.` When X is sober, Exercise 15.10 shows that X can be entirely constructed from its locale $\mathrm{O}(X)$.

`Construction 15.12.` Given a locale L there is a locale morphism $\epsilon_L \colon \mathrm{O}(|L|) \to L$. The corresponding poset morphism $\epsilon_L^* \colon L \to \mathrm{O}(|L|)$ takes $s \in L$ to $|L|_s$. Given a morphism of locales $f \colon L \to M$ corresponding to a map of posets $f^* \colon M \to L$, we claim that

$$\begin{tikzcd} \mathrm{O}(|L|) \arrow[r, "\epsilon_L"] \arrow[d, "\mathrm{O}(|f|)"'] & L \arrow[d, "f"] \\ \mathrm{O}(|M|) \arrow[r, "\epsilon_M"'] & M \end{tikzcd}$$

commutes in **Loc**. This amounts to the square

$$\begin{tikzcd} M \arrow[r, "\epsilon_M^*"] \arrow[d, "f^*"'] & \mathrm{O}(|M|) \arrow[d, "\mathrm{O}(|f|)^*"'] \\ L \arrow[r, "\epsilon_L^*"'] & \mathrm{O}(|L|) \end{tikzcd}$$

commuting. Let $x \in M$. Then, $\epsilon_L^*(f^*(x)) = |L|_{f^*(x)}$. The other way around the square produces $\mathrm{O}(|f|)^*(\epsilon_M^*(x)) = \mathrm{O}(|f|)^*(|M|_x)$. These are both subsets of $|L|$. A point $y \in |L|$ is in $|L|_{f^*(x)}$ if and only if $f^*(x) \nleq y$. A point is in $\mathrm{O}(|f|)^*(|M|_x)$ if and only if $|f|(y) \in |M|_x$ if and only if $x \nleq |f|(y)$. These two agree by Remark 14.17. It follows that the ϵ_L assembly into a natural transformation $\epsilon \colon \mathrm{O} \circ |-| \to \mathrm{id}_{\mathbf{Loc}}$.

`Definition 15.13 (Adjunctions).` Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor, and let $\mathcal{C} \leftarrow \mathcal{D} \colon G$ be a functor. An adjunction between F and G consists of natural transformations $\eta \colon \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon \colon F \circ G \to \mathrm{id}_{\mathcal{D}}$ (called the unit and counit of the adjunction) such that the following conditions hold:

(i) for each $c \in \mathcal{C}$, the composition

$$F(c) \xrightarrow{F(\eta_c)} F(G(F(c))) \xrightarrow{\epsilon_{F(c)}} F(c)$$

is $\mathrm{id}_{F(c)}$;

(ii) for each $d \in \mathcal{D}$, the composition

$$G(d) \xrightarrow{\eta_{G(d)}} G(F(G(d))) \xrightarrow{G(\epsilon_d)} G(d)$$

is $\mathrm{id}_{G(d)}$.

In this case, we say that F is left adjoint to G and G is right adjoint to F. It is displayed as

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G.$$

`Remark 15.14.` Any time F and G are inverse equivalences of categories, they are adjoint (exercise!), but adjunctions exist in much greater abundance than equivalences, and are much more important.

`Construction 15.15.` Suppose that $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ is an adjunction (so that η and ϵ are implicit). Let $c \in \mathcal{C}$ and $d \in \mathcal{D}$. There is a canonical function $\mathrm{Hom}_{\mathcal{C}}(c, G(d)) \to \mathrm{Hom}_{\mathcal{D}}(F(c), d)$ constructed as the composition

$$\mathrm{Hom}_{\mathcal{C}}(c, G(d)) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(c), F(G(d))) \xrightarrow{\epsilon_d} \mathrm{Hom}_{\mathcal{D}}(F(c), d).$$

We also have a function in the other direction given by the composition

$$\mathrm{Hom}_{\mathcal{D}}(F(c), d) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(G(F(c)), G(d)) \xrightarrow{\eta_c} \mathrm{Hom}_{\mathcal{C}}(c, G(d)).$$

`Exercise 15.16.` Use the definition of an adjunction to show that the maps of Construction 15.15 are inverse equivalences.

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