Category, space, type - Benjamin Antieau
14. Locales I

We have seen something about the theory of lattices in the previous lecture. A bounded lattice bears a striking resemblance to the set of opens in a topological space X: there are finite meets and joins (like unions and intersections) as well as initial and terminal objects (like $\emptyset$ and X). In this section, we introduce the notion of a locale based on the axioms the open subsets of a topological space satisfy. Having done this, it makes sense to ask two questions. To what extent is a space determined by its associated locale? Does every locale come from a topological space?

Lemma 14.1. *Let P be a poset. If every subset $S \subseteq P$ has a least upper bound $\vee S \in P$, then every subset $S \subseteq P$ has a greatest lower bound $\wedge S \in P$.*

Proof. Fix $S \subseteq P$. Let $T \subseteq P$ be the set of elements x such that $x \leqslant y$ for all $y \in S$. Thus, T is the set of lower bounds of S. Let $t \in P$ be the least upper bound of T. Note that every element of S is an upper bound of T. Thus, t is less than or equal to every element of S; thus, $t \in T$ and it is a greatest lower bound of S. $\square$

Remark 14.2. Suppose that P is a poset in which every subset has a least upper bound. We will write $x \vee y$ for $\vee\{x, y\}$ and $x \wedge y$ for $\wedge\{x, y\}$. The poset P also admits a least element $\bot$ and a greatest element $\top$. These are the least upper bound and greatest lower bound of $\emptyset$.

Remark 14.3. We call $\vee S$ the join of S and $\wedge S$ the meet of S.

Definition 14.4. A poset P is a locale if it satisfies the following two conditions:

(i) every subset $S \subseteq P$ has a least upper bound $\vee S \in P$;

(ii) finite meets distribute over joins: if $x \in P$ and $S \subseteq P$, then

$$x \wedge (\vee S) = \vee_{s \in S}(x \wedge s).$$

Example 14.5. A locale is a distributive lattice. A finite distributive lattice is a locale.

`Example 14.6.` Let $(X, \mathcal{U})$ be a topological space. Viewed as a partial set with respect to inclusion, $\mathcal{U}$ is a locale. Axiom (i) is satisfied because any set of open subsets of X has a greatest lower bound, namely their union. Axiom (ii) is satisfied by the usual rules of set theory.

`Remark 14.7.` Thinking about it in these terms, the reader might be surprised by Lemma 14.1, which states that every collection of open subsets of a topological space has a greatest lower bound. However, in general, this greatest lower bound corresponds not to the intersection but to the interior of the intersection.

`Definition 14.8.` Let L and M be locales. A morphism of locales from L to M is a function $f^*\colon M \to L$ which preserves joins and finite meets. Specifically, if $S \subseteq M$, then $f^*(\vee S) = \vee f^*(S)$ and if additionally $S \subseteq M$ is finite, then $f^*(\wedge S) = \wedge f^*(S)$.

`Definition 14.9 (Category of locales).` With the notion of morphism of locales above, we obtain the category **Loc** of locales: objects are locales and morphisms are morphisms of locales.

`Example 14.10.` Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be topological spaces. Let $f\colon X \to Y$ be a continuous function. Then, $f^{-1}\colon \mathcal{V} \to \mathcal{U}$ defines a morphism of locales $\mathcal{U} \to \mathcal{V}$. It follows, that there is a functor $\mathrm{O}\colon \mathbf{Top} \to \mathbf{Loc}$. This functor is definitely not an equivalence. For example, the simplest locale is the poset

$$\bot \to \top.$$

If we take X^{triv}, a set X equipped with the trivial topology, then $\mathrm{O}(X^{\mathrm{triv}})$ is isomorphic to $\bot \to \top$. Nevertheless, we will see that there is an equivalence of categories once we impose a restriction on the topological spaces in question. To motivate this, we first consider the problem of constructing a functor $\mathbf{Loc} \to \mathbf{Top}$.

`Definition 14.11 (Indecomposables).` Let L be a locale. A point $x \in L$ is indecomposable if the following condition holds: for every finite subset $S \subseteq L$, if $x = \wedge S$, then $x \in S$. Let $|L|$ denote the set of indecomposable elements of L.

`Example 14.12.` Let $L = \{\bot \to \top\}$ be the Boolean locale. Then, $|L| = \{\bot\}$. Note that $\top = \wedge\emptyset$, so $\top$ is not indecomposable.

`Definition 14.13 (Opens).` If L is a locale and $s \in L$, let $|L|_s = \{x \in |L| : s \nleqslant x\}$.

`Lemma 14.14.` *Let L be a locale and $S \subseteq L$ a subset.*

(a) *We have $|L|_{\vee S} = \bigcup_{s \in S} |L|_s$.*

(b) *If S is finite, then $|L|_{\wedge S} = \bigcap_{s \in S} |L|_s$.*

Proof. For $s \in S$ we have $s \leqslant \vee S$ which implies by transitivity that if $x \nleqslant s$, then $x \nleqslant \vee S$. Thus, the right-hand side of (a) is contained in the left. Suppose that $x \in |L|_{\vee S}$ so that $\vee S \nleqslant x$. If $s \leqslant x$ for every $s \in S$, then x is an upper bound for S so that $\vee S \leqslant x$ since $\vee S$ is the least upper bound for S. Thus, there is some $s \in S$ such that $s \nleqslant x$ and hence $x \in |L|_s$. Hence, the left-hand side is contained in the right and (a) is proved.

For (b), we have $\wedge S \leqslant s$ for each $s \in S$. Thus, if $\wedge S \nleqslant x$, then $s \nleqslant x$ for each $s \in S$. Hence, we have $|L|_{\wedge S} \subseteq \bigcap_{s \in S} |L|_s$. On the other hand, if $s \nleqslant x$ for each $s \in S$, then $x < x \vee s$ for each $s \in S$. If $\wedge s \leqslant x$, then we have $x = x \vee (\wedge s) = \wedge_{s \in S}(x \vee s)$, which contradicts indecomposability of x. □

It follows that if L is a locale, then the subsets of $|L|$ of the form $|L|_s$ for $s \in L$ form a topology: part (a) shows they are closed under arbitrary unions and part (b) shows that they are closed under finite intersections. The empty set is $|L|_\bot$ and $|L| = |L|_\top$, the latter since $\top$ itself is not indecomposable.

`Construction 14.15.` Show that the points of $|L|$ are in bijection with the set of locale morphisms $\{\bot \to \top\} \to L$. Specifically, given $f^* \colon L \to \{\bot \to \top\}$, one can take $(f^*)^{-1}(\bot)$. Let x be the least upper bound of this set. We claim that x is indecomposable. If $x = \wedge S$, then $x \leqslant s$ for $s \in S$. If $x < s$, then we must have $f^*(s) = \top$. If S is finite, then $f^*(\wedge S) = \wedge_{s \in S} f^*(s) = \top$. It follows that x is indecomposable. Note also that by compatibility of f^* with arbitrary joins we have $f^*(x) = \bot$.

`Exercise 14.16.` Suppose that $f \colon L \to M$ is a morphism of locales (corresponding to a morphism of posets $f^* \colon M \to L$). We would like to define a continuous function $|f| \colon |L| \to |M|$ associated to f. To do so, we note that if $x \in |L|$ corresponds to a locale morphism $\{\bot \to \top\} \xrightarrow{x} L$, then we can compose with f to obtain a locale morphism $\{\bot \to \top\} \xrightarrow{x} L \xrightarrow{f} M$ to obtain a point of M. This defines $|f|$. Show that $|f|$ is continuous. Show that this assignment $f \mapsto |f|$ defines a functor $|-|\colon \mathbf{Loc} \to \mathbf{Top}$.

Remark 14.17. If $f\colon L \to M$ is a morphism of locales with corresponding morphism of posets $f^*\colon M \to L$, then $|f|$ is characterized by the following property: $f^*(x) \nleq y$ if and only $x \nleq |f|(y)$ for $x \in M$ and $y \in |L|$. Indeed, y corresponds to a function, say $y^*\colon L \to \{\bot, \top\}$ with $y^*(a) = \top$ if and only if $a \nleq y$. We defined $|f|(y)$ as the indecomposable associated to the composition $|f|(y)^*\colon M \xrightarrow{f^*} L \xrightarrow{y^*}$. Thus, $|f|(y)^*(b) = \top$ if and only if $b \nleq |f|(y)$ if and only if $f(b) \nleq y$.

Construction 14.18. Given a topological space X there is a continuous function $\eta_X\colon X \to |\mathrm{O}(X)|$. To construct it, we must assign to any $x \in X$ an indecomposable of $\mathrm{O}(X)$. But, we can view x as a continuous function $* \xrightarrow{x} X$ which corresponds to a morphism of locales $\{\bot \to \top\} \to \mathrm{O}(X)$ by functoriality; we have already observed that these correspond to indecomposables of $\mathrm{O}(X)$. Concretely, to x, we assign the interior of $X \setminus \{x\}$. The reader can check that if $f\colon X \to Y$ is a continuous function, then

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & |\mathrm{O}(X)| \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle |\mathrm{O}(f)|} \\ Y & \xrightarrow[\eta_Y]{} & |\mathrm{O}(Y)| \end{array}$$

commutes. In other words, the η_X assemble to define a natural transformation $\eta\colon \mathrm{id}_{\mathbf{Top}} \to |-| \circ \mathrm{O}$.

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