Category, space, type - Benjamin Antieau

13. Birkhoff's representation theorem

The following is a nice example of a non-trivial duality arising in the topology of finite spaces. Recall that a topological space is finite if it has a finite number of points. A poset is finite if it has a finite number of points. Every finite topological space is Alexandrov. Thus, our equivalence (in fact isomorphism)

$$\mathbf{Pos} \simeq \mathbf{Alexandrov}^{T_0}$$

between posets and T_0 Alexandrov restricts to an equivalence

$$\mathbf{Pos}^{\mathrm{fin}} \simeq \mathbf{Alexandrov}^{T_0,\mathrm{fin}} \simeq \mathbf{Top}^{T_0,\mathrm{fin}}$$

between finite posets and finite T_0 topological spaces.

Finite posets are a basic object in combinatorics. The reader can look up the number of isomorphism classes of finite posets on n elements in `The on-line encyclopedia of integer sequences`, also called the `OEIS`. Specifically, `OEIS: A000112` gives the page for this sequence. In the table below, the first row is the number of vertices and the second row gives the number of isomorphism classes of posets with that number of vertices. The largest number known

0	1	2	3	4	5	6	7	8	9	10
1	1	2	5	16	63	318	2045	16999	183231	2567284

for this sequence is for 16: there are 4483130665195087 isomorphism classes of posets with 16 elements. Unsurprisingly, these are tabulated by a computer search. We will see another incarnation of the category of finite posets below.

`Definition 13.1 (Join and meet).` Let P be a poset and let $x, y \in P$. The join of x and y, if it exists, is the least upper bound of x and y in P. It is written as $x \vee y$. Similarly, the meet of x and y, if it exists is the greatest lower bound of x and y. Write it as $x \wedge y$.

`Remark 13.2.` More generally, we can speak of the join or meet of any subset of a poset.

`Example 13.3.` Consider the following poset on four elements w, x, y, z.

$$\begin{tikzcd}
w \arrow[r] \arrow[dr] & y \\
x \arrow[r] \arrow[ur] & z.
\end{tikzcd}$$

The pair $\{w, x\}$ has neither a join $w \vee x$ nor a meet $w \wedge x$.

`Example 13.4.` Consider the following poset on four elements w, x, y, z.

$$\begin{tikzcd} w \arrow[r] & y \\ x \arrow[r] \arrow[ur] & z. \end{tikzcd}$$

In this poset, $w \vee x = y$ and $y \wedge z = x$. However, $\{w, x\}$ does not have a meet and $\{y, z\}$ does not have a join.

`Definition 13.5 (Lattices).` A lattice is a poset P such that for every pair of elements $x, y \in P$ there is a join $x \vee y$ and a meet $x \wedge y$ in P.

`Definition 13.6 (Bounded lattices).` A lattice L is bounded if it has a minimal element, often written as $\emptyset$ or $\perp$, and a maximal element, often written as $*$ or $\top$.

`Remark 13.7.` If they exist, $\perp$ is the join of the empty set of elements of L in the sense that $\perp$ is the least upper bound of the elements of $\emptyset$. Similarly, $\top$ is the greatest lower bound of the empty set of elements of L.

`Exercise 13.8.` Show that any finite lattice is bounded.

`Example 13.9.` Here are a few bounded lattices. Of course, the poset on 1 point is a bounded lattice. There is only one bounded lattice on 2 points:

$$\perp \longrightarrow \top.$$

On 3 points there is also only one:

$$\perp \longrightarrow x \longrightarrow \top.$$

On 4 points there are a couple options. There is the linear option

$$\perp \longrightarrow x \longrightarrow y \longrightarrow \top$$

and a new option

$$\begin{tikzcd} & x \arrow[dr] & \\ \perp \arrow[ur] \arrow[dr] & & \top. \\ & y \arrow[ur] & \end{tikzcd}$$

`Exercise 13.10.` Find all isomorphism classes of bounded lattices with 5 vertices.

`Definition 13.11 (Distributive lattices).` A lattice L is distributive if meets distribute over joins in the sense that for elements x, y, z in L we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

`Exercise 13.12.` Prove that a lattice L is distributive if and only if joins distribute over meets: for every w, x, y we have

$$w \vee (x \wedge y) = (w \vee x) \wedge (w \vee y).$$

`Example 13.13.` The examples of lattices in Example 13.9 are all distributive. The lattice

$$\begin{tikzcd}
& x \arrow[dr] & \\
\perp \arrow[ur] \arrow[r] \arrow[dr] & y \arrow[r] & \top \\
& z \arrow[ur] &
\end{tikzcd}$$

is not distributive. Indeed, $x \wedge (y \vee z) = x \wedge \top = x$ but $(x \wedge y) \vee (x \wedge z) = \perp \vee \perp = \perp$.

`Example 13.14.` If X is a set, then the power set $\mathbf{P}(X)$, viewed as a poset with respect to inclusion of subsets, is a bounded distributive lattice.

`Example 13.15.` Let $n \geqslant 1$ be a natural number. Let D be the set of positive divisors of n viewed as a poset with respect to divisibility. Then, D is a distributive lattice, necessarily bounded because it is finite. For example, if $n = 12$, then $D = \{1, 2, 3, 4, 6, 12\}$ with poset structure given by

$$\begin{tikzcd}
& 2 \arrow[r] \arrow[ddr] & 4 \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & & 12. \\
& 3 \arrow[r] & 6 \arrow[ur] &
\end{tikzcd}$$

`Definition 13.16 (Join-irreducibles).` Let L be a lattice. We say that an object $x \in L$ is join-irreducible if x is not the join of a finite set of objects of L not containing itself. In particular, the initial object $\perp$, if it exists in L, is not join-irreducible as it is the join of the empty set of objects. The set of join-irreducibles inherits a poset structure from L and is called the spectrum of L and written as $\operatorname{Spec} L$.

`Example 13.17.` In Example 13.15, the join-irreducible elements of the divisibility lattice associated to 12 are the elements $2, 3, 4$. These form the disconnected poset

$$\begin{tikzcd} 2 \arrow[r] & 4. \\ 3 & \end{tikzcd}$$

`Definition 13.18 (Lattice homomorphisms).` Let L and M be bounded distrbutive lattices. A bounded lattice homomorphism is a function $f \colon L \to M$ such that $f(\perp) = \perp$, $f(\top) = \top$, $f(x \vee y) = f(x) \vee f(y)$, and $f(x \wedge y) = f(x) \wedge f(y)$ for all pairs $x, y \in L$.

`Remark 13.19.` One could rephrase the definition as saying that f commutes with finite meets and joins. Preserving the initial and final objects then follows from preserving empty meets and joins.

`Exercise 13.20.` Show that if f is a bounded lattice homomorphism, then f is a morphism on underlying posets: if $x \leqslant y$, then $f(x) \leqslant f(y)$.

`Definition 13.21.` Let $\mathbf{Dist}^{\mathrm{fin}}$ be the category of finite distributive lattices and bounded lattice homomorphisms between them.

`Theorem 13.22 (Birkhoff).` *The spectrum construction induces an equivalence* $\operatorname{Spec} \colon \mathbf{Dist}^{\mathrm{fin}} \simeq \mathbf{Pos}^{\mathrm{fin,op}}$. *In other words,* Spec *induces a duality between* $\mathbf{Dist}^{\mathrm{fin}}$ *and* $\mathbf{Pos}^{\mathrm{fin}}$.

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