Category, space, type - Benjamin Antieau

12. Equivalences of categories

We discuss in this section the notion of ‘sameness’ for categories. On the one hand, we have the innate notion of isomorphism in the category $\mathcal{C}\mathrm{at}$ of categories.a This notion is very strict in the sense that if $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are inverse isomorphisms of categories, then $G \circ F$ is *equal* to the identity functor on $\mathcal{C}$. In particular, $G(F(c)) = c$ for every object $c \in \mathcal{C}$. Very frequently what happens instead is that one can find F and G such that $G(F(c))$ is isomorphic to c, but not equal to it. This leads to the notion of natural transformations of functors and to the notion of equivalences of categories.

`Definition 12.1 (Natural transformations).` Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation from F to G, written for example as $F \xrightarrow{\eta} G$, consists of a morphism $\eta_c\colon F(c) \to G(c)$ for each $c \in \mathcal{C}$ such that for each $f\colon c \to c'$ in $\mathcal{C}$ the diagrams

$$\begin{array}{ccc} F(c) & \xrightarrow{\eta_c} & G(c) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow[\eta_{c'}]{} & G(c') \end{array} \qquad (1)$$

commute.

`Remark 12.2 (Recollection on commutative diagrams).` A square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle i} \\ c & \xrightarrow[g]{} & d \end{array}$$

of objects and morphisms in a category $\mathcal{C}$ commutes if $g \circ h = i \circ f$.

`Exercise 12.3.` Given natural transformations $\eta\colon F \to G$ and $\epsilon\colon G \to H$ of functors $F, G, H\colon \mathcal{C} \to \mathcal{D}$, show that there is a composite natural transformation $\epsilon \circ \eta\colon F \to H$.

`Exercise 12.4.` Let $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}, \mathcal{D})$ be the set of functors from $\mathcal{C}$ to $\mathcal{D}$. Show that this set can be made into the set of objects of a category, which we will denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The objects are functors and the morphisms are natural transformations. Composition is given by your solution to Exercise 12.3.

`Remark 12.5.` Categories fit into a natural heirarchy of more and more complicated objects called n-categories. These have a notion of k-morphism for $1 \leqslant k \leqslant n$. In this language, 0-categories are sets and 1-categories are categories as we have discussed previously. The first example of a 2-category is an analogue of $\mathcal{C}\mathrm{at}$. Let's call it $\mathcal{C}\mathrm{at}_1$ to disambiguate it from $\mathcal{C}\mathrm{at}$ for the moment. The objects of $\mathcal{C}\mathrm{at}_1$ are categories, the 1-morphisms are functors, and the 2-morphisms are natural transformations between functors. This is the beginning of a deep and beautiful subject, but we will not pursue it here.

`Example 12.6.` Recall that given a topological space X we have a poset $\mathrm{P}(X)$. Given a poset P, we obtain a topological space $\mathrm{D}(P)$ by taking the topology given by the downsets. We also observed in Theorem 10.1(a) that there is a continuous function $c_X \colon \mathrm{D}(\mathrm{P}(X)) \to X$. In fact, this defines a natural transformation $c \colon \mathrm{D} \circ \mathrm{P} \to \mathrm{id}_{\mathbf{Top}}$. To check this, for each continuous function $f \colon X \to Y$, we must check that the diagram

$$\begin{array}{ccc} \mathrm{D}(\mathrm{P}(X)) & \xrightarrow{c_X} & X \\ {\scriptstyle \mathrm{D}(\mathrm{P}(f))}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{D}(\mathrm{P}(Y)) & \xrightarrow[c_Y]{} & Y \end{array}$$

commutes. However, on underlying sets, c_X and c_Y are the identities and $\mathrm{D}(\mathrm{P}(f)) = f$ by Theorem 10.1(d,e), so the diagram does indeed commute.

`Definition 12.7 (Natural isomorphisms).` Let $\eta \colon F \to G$ be a natural transformation of functors from $\mathcal{C}$ to $\mathcal{D}$. If for each $c \in C$ the morphism $\eta_c \colon F(c) \to G(c)$ is an isomorphism in $\mathcal{D}$, then we say that η is a natural isomorphism. We say that F and G are naturally isomorphic if there exists a natural isomorphism η from F to G.

`Exercise 12.8.` Show that η is a natural isomorphism if and only if it is an isomorphism when viewed as a morphism in the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

`Definition 12.9 (Equivalences of categories).` Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is an equivalence of categories if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ is naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $F \circ G$ is naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$. We say that $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists an equivalence between them. I tend to write $\mathcal{C} \simeq \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are equivalent and reserve $\cong$ as the generic symbol for isomorphism.

`Exercise 12.10.` Show that $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is fully faithful and essentially surjective.

`Example 12.11.` Let $\mathbf{Vect}_{\mathbf{R}}^{\mathrm{fd}}$ be the category of finite-dimensional $\mathbf{R}$-vectors spaces and linear transformations between them. Let $\mathcal{V}_{\mathbf{R}}$ be the category with objects $\mathbf{R}^0, \mathbf{R}^1, \mathbf{R}^2, \ldots$ and where $\mathrm{Hom}_{\mathcal{V}_{\mathbf{R}}}(\mathbf{R}^n, \mathbf{R}^m)$ is equal to the set of $m \times n$-matrices with real entries. There is a fully faithful, essentially surjective functor $\mathcal{V}_{\mathbf{R}} \to \mathbf{Vect}_{\mathbf{R}}^{\mathrm{fd}}$, so these categories are equivalent. They are not isomorphic.

`Definition 12.12 (Opposite category).` Let $\mathcal{C}$ be a category. There is another category $\mathcal{C}^{\mathrm{op}}$ with the same objects as $\mathcal{C}$ but where $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(y, x)$. It is obtained by "turning all arrows around".

`Definition 12.13.` A duality between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence $\mathcal{C} \simeq \mathcal{D}^{\mathrm{op}}$. If $\mathcal{C} \simeq \mathcal{D}^{\mathrm{op}}$, then $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{D}$.

There are many famous equivalences of categories in mathematics. Many of these, especially in topology, take the form of a duality between a "topological" category and an "algebraic" one. Here are a few.

(a) The duality between sober topological spaces and locales.

(b) The equivalence between posets and Alexandrov topological spaces.

(c) Stone duality: the duality between Boolean algebras and completely disconnected compact Hausdorff spaces.

(d) Gelfand duality: the duality between compact Hausdorff topological spaces and unital commutative C^*-algebras.

(e) The Birkhoff representation theorem, a equivalence between the category of finite distributive lattices and bounded homomorphisms and the category of finite posets and order-preserving morphisms.

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