`Category, space, type - Benjamin Antieau`
`11. Functors`

Just as we organize other kinds of mathematical objects into categories, similarly we organize categories themselves into a category, denoted $\mathcal{C}\mathrm{at}$ below. The set of objects of $\mathcal{C}\mathrm{at}$ is the class of categories. The morphisms in $\mathcal{C}\mathrm{at}$ are called *functors* and are the subject of this lecture.

`Definition 11.1 (Functors).` Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of

(a) a function $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$

(b) and, for each pair $x, y \in \mathcal{C}$, a function $F\colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$

subject to the conditions that

(i) for each $x \in \mathcal{C}$ we should have $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ and

(ii) for each pair of morphisms $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$ in $\mathcal{C}$ we should have $F(g \circ f) = F(g) \circ F(f)$ in $\mathrm{Hom}_{\mathcal{C}}(F(x), F(z))$.

`Remark 11.2.` In order to avoid slavish repitition we do not often distinguish between a functor, its action on objects, and its action on morphism. One could think of this as being defined by "overloading operators" as in a programming language. The same symbol '+' might define addition between vectors, matrices, real numbers,... We could unpack this by defining a functor as a pair

$$F = (F_{\mathrm{Ob}}, \{F_{(x,y)}\}_{(x,y)\in \mathrm{Ob}(\mathcal{C})^2}),$$

where $F_{\mathrm{Ob}}\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ is a function and where $F_{(x,y)}\colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F_{\mathrm{Ob}(\mathcal{C})}(x), F_{\mathrm{Ob}(\mathcal{C})}(y))$ is a function for each $(x, y) \in \mathrm{Ob}(\mathcal{C})^2$, and which satisfies (i) and (ii) above.

`Example 11.3.` For each category $\mathcal{C}$ there is an identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$, obtained by taking the appropriate identites on object and morphism sets. Specifically, the identity functor is the identity *function* on $\mathrm{Ob}(\mathcal{C})$ and for each $x, y \in \mathcal{C}$ the identity function on $\mathrm{Hom}_{\mathcal{C}}(x, y)$.

`Definition 11.4 (The category of posets).` Let **Pos** be the category of posets: the objects of **Pos** are posets and, if P, Q are posets, $\mathrm{Hom}_{\mathbf{Pos}}(P, Q)$ is the set of functions $f\colon P \to Q$ such that if $x \leqslant y$ in P, then $f(x) \leqslant f(y)$ in Q.

`Definition 11.5 (The category of preorders).` Let **PreOrd** be the category of preorders: the objects of **PreOrd** are preorders, i.e., sets P equipped with a preorder $\lesssim$, a reflexive and transitive binary relation. If P, Q are preorders, then $\mathrm{Hom}_{\mathbf{PreOrd}}(P, Q)$ is the set of functions $f\colon P \to Q$ such that if $x \lesssim y$ in P, then $f(x) \lesssim f(y)$ in Q.

`Definition 11.6 (Full subcategories).` Let $\mathcal{C}$ be a category. Fix a subset of objects $S \subseteq \mathrm{Ob}(\mathcal{C})$. We can define a new category $\mathcal{D}$ where $\mathrm{Ob}(\mathcal{D}) = S$ and where if $x, y \in S$, then $\mathrm{Hom}_{\mathcal{D}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y)$. Identities and compositions are also defined using $\mathcal{C}$. The inclusion $S \subseteq \mathrm{Ob}(\mathcal{C})$ allows us to define an inclusion functor $\mathcal{D} \to \mathcal{C}$ by sending an object of $\mathcal{D}$ to its corresponding object in $\mathcal{C}$ via $S \subseteq \mathrm{Ob}(\mathcal{C})$.

`Definition 11.7.` More generally, if $\mathcal{C}$ is a category, we can define a (possibly non-full subcategory) as in the previous definition, but where for each $x, y \in S$ we have that $\mathrm{Hom}_{\mathcal{D}}(x, y)$ is a *subset* of $\mathrm{Hom}_{\mathcal{C}}(x, y)$. We require $\mathrm{Hom}_{\mathcal{D}}(x, x)$ to contain id_x (with respect to $\mathcal{C}$) and for this collection of subsets to be closed under composition in that if $f \in \mathrm{Hom}_{\mathcal{D}}(x, y)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(y, z)$, then $g \circ f$ (computed in $\mathcal{C}$) is in $\mathrm{Hom}_{\mathcal{D}}(x, z)$.

`Example 11.8.` We can view **Pos** as a full subcategory of **PreOrd**.

`Example 11.9.` We can let $\mathbf{Top}^{T_0} \subseteq \mathbf{Top}$ be the full subcategory of T_0 topological spaces.

`Example 11.10.` Let $\mathbf{Alexandrov} \subseteq \mathbf{Top}$ be the full subcategory of Alexandrov spaces. The category $\mathbf{Alexandrov}^{T_0}$ of T_0 Alexnadrov spaces is a full subcategory of **Top**, $\mathbf{Top}^{T_0}$, and $\mathbf{Alexandrov}^{T_0}$.

`Definition 11.11 (Fully faithful functors).` A functor $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful if for every pair $x, y \in \mathcal{C}$ the function $F\colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ is a bijection.

`Example 11.12.` If $\mathcal{D}$ is a full subcategory of $\mathcal{C}$, then the canonical inclusion functor $\mathcal{D} \to \mathcal{C}$ is fully faithful.

`Definition 11.13 (Essentially surjective functors).` A functor $F\colon \mathcal{C} \to \mathcal{D}$ is essentially surjective if for every object $d \in \mathcal{D}$ there is an object $c \in \mathcal{C}$ and an isomorphism $F(c) \cong d$ in $\mathcal{D}$.

`Definition 11.14.` Essential image Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, the essential image of F is the full subcategory of $\mathcal{D}$ consisting of those objects $d \in \mathcal{D}$ such that there exists $c \in \mathcal{C}$ such that $F(c) \cong d$.

`Remark 11.15.` We could also speak of the image of a functor $F\colon \mathcal{C} \to \mathcal{D}$, as opposed to the essential image. This would be the full subcategory of objects $d \in \mathcal{D}$ *equal* to $F(c)$ for some $c \in \mathcal{C}$. However, the notion of essential image (and of essentially surjective) is more robust and more useful. A subcategory $\mathcal{E} \subseteq \mathcal{C}$ is *replete* if whenever $e \in \mathcal{E}$ and $c \in \mathcal{C}$ is isomorphic to e, then $c \in \mathcal{E}$.

`Definition 11.16 (Specialization preorder functor).` Given each topological space $(X, \mathcal{U})$ we have defined the specialization preorder $\lesssim_{\mathcal{U}}$ on the set of points X. This defines a functor

$$\mathbf{Top} \to \mathbf{PreOrd}.$$

`Definition 11.17 (Specialization partial order functor).` Restricting the specialization preorder functor to T_0 toplogical spaces, we can obtain a specialization poset functor

$$\mathrm{P}\colon \mathbf{Top}^{T_0} \to \mathbf{Pos}.$$

The content of this is that not only if $\mathrm{P}(X)$ a poset if X is a topological space, but if $f\colon X \to Y$ is a continuous function, then $\mathrm{P}(f)\colon \mathrm{P}(X) \to \mathrm{P}(Y)$ is a map of posets. (See Theorem 10.1(d).)

`Construction 11.18.` We have also described a functor

$$\mathrm{D}\colon \mathbf{Pos} \to \mathbf{Top}^{T_0}$$

called the topological space associated to a poset which uses the downsets as the opens. If $g\colon P \to Q$ is a map of posets, then $\mathrm{D}(g)\colon \mathrm{D}(P) \to \mathrm{D}(Q)$ is a continuous function. (See Theorem 10.1(e).)

`Construction 11.19 (Functor composition).` Given categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, as well as functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ there is a composite functor $G \circ F\colon \mathcal{C} \to \mathcal{E}$. On object sets it is the composite

$$\mathrm{Ob}(\mathcal{C}) \xrightarrow{F} \mathrm{Ob}(\mathcal{D}) \xrightarrow{G} \mathrm{Ob}(\mathcal{E}).$$

For $x, y \in \mathcal{C}$, it is given as the composition

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y)) \to \mathrm{Hom}_{\mathcal{E}}(G(F(x)), G(F(y))).$$

`Exercise 11.20.` Show that if $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$, and $H\colon \mathcal{E} \to \mathcal{F}$ are functors, then $H \circ (G \circ F) = (H \circ G) \circ F$.

Thus, functor composition is associative.

`Remark 11.21.` Theorem 10.1 can be summarized as saying that D is fully faithful with essential image $\mathbf{Alexandrov}^{T_0}$. More specifically, Theorem 10.1(c) implies that

$$\mathrm{P} \circ \mathrm{D} = \mathrm{id}_{\mathbf{Pos}}.$$

We do not have $\mathrm{D} \circ \mathrm{P} = \mathrm{id}_{\mathbf{Top}^{T_0}}$. However, we do have $\mathrm{P} \circ \mathrm{D} \circ \mathrm{P} = \mathrm{P}$. If we let $\mathrm{D}'\colon \mathbf{Pos} \to \mathbf{Alexandrov}^{T_0}$ be the downset topology functor, but viewed as having codomain $\mathbf{Alexandrov}^{T_0}$, and if we let $\mathrm{P}'\colon \mathbf{Alexandrov}^{T_0} \to \mathbf{Pos}$ be the specialization poset functor, but viewed as having domain $\mathbf{Alexandrov}^{T_0}$, then we do have

$$\mathrm{D}' \circ \mathrm{P}' = \mathrm{id}_{\mathbf{Top}^{T_0}}$$

by Theorem 10.1(b).

`Definition 11.22 (The category of categories).` We ignore set theory, cardinality issues, and Russel's paradox and define the category $\mathcal{C}$at to have as objects categories and to have as morphisms the functors, so $\mathrm{Hom}_{\mathcal{C}}\mathrm{at}(\mathcal{C}, \mathcal{D})$ is the set of functors from $\mathcal{C}$ to $\mathcal{D}$.

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