Category, space, type - Benjamin Antieau
09. Order

Last time, we introduced the notion of compactness. The next theorem is familiar from analysis.

Theorem 9.1 (Heine-Borel). *If $a \leqslant b \in \mathbf{R}$, then the interval $[a,b]$ is compact.*

Recall the usual proof, which uses the crucial property of the real numbers that each nonempty bounded above subset has a least upper bound.

Proof. Let $\mathcal{C}$ be an open cover of $[a,b]$. Consider the set $A \subseteq [a,b]$ such that $x \in A$ if and only if there is a finite subcover $\mathcal{D}_x \subseteq \mathcal{C}$ such that $\mathcal{D}_x$ covers $[a,x]$. Since $\mathcal{C}$ is a cover of $[a,b]$, we have $a \in U$ for some $U \in \mathcal{C}$, which shows that $a \in A$ and in fact $[a, a+\delta) \subseteq A$ for some $\delta > 0$. On the other hand, b is an upper bound for A. Let c be the least upper bound of A. Thus, $a \leqslant c \leqslant b$. Let $V \in \mathcal{C}$ be an open which contains c. Then, V contains $(c-\epsilon, c]$ for some $\epsilon > 0$. By assumption, if $d \in (c-\epsilon, c)$, then there is a finite subcover $\mathcal{D}_d \subseteq \mathcal{C}$ such that $\mathcal{D}_d$ covers $[a,d]$. But, then, $\{V\} \cup \mathcal{D}_d$ covers $[a,c]$. It follows that $c \in A$. If $c < b$, then we can assume ϵ is so small that $(c-\epsilon, c+\epsilon) \subseteq V$ and $c+\epsilon < b$. But, then $c + \frac{\epsilon}{2} \in A$, a contradiction. □

Corollary 9.2. *A subset $A \subseteq \mathbf{R}$ is compact if and only if it is closed and bounded.*

Proof. If A is compact, then it is closed since $\mathbf{R}$ is Hausdorff; see Corollary 7.15. But, it is also bounded since a finite collection of intervals of the form $(n, n+2)$, for $n \in \mathbf{Z}$, must cover A. Conversely, a bounded closed subset is in particular a subset of $[-N, N]$ for some N which is compact. As a closed subset of a compact space, it is compact. □

Corollary 9.3. *A continuous bijection $f\colon \mathbf{R} \to \mathbf{R}$ is a homeomorphism.*

Proof. The function f must be mononotonic, i.e., either $f(x) < f(y)$ if $x < y$ or $f(x) > f(y)$ if $x < y$. Indeed, otherwise the intermediate value theorem would show that f is not injective. Consider a closed interval $[a,b] \subseteq \mathbf{R}$. It is compact by the Heine–Borel theorem. Thus,

$f([a,b]) \subseteq \mathbf{R}$ is compact. Thus, it is closed. We also have that $f([a,\infty))$ is either $[f(a),\infty)$ or $[-\infty, f(a))$. These are closed in either case. Hence, f is a closed map. Since it is a bijection, it is an open map too. □

`Remark 9.4.` This extends Theorem 7.17, which says that if $f\colon X \to Y$ is a continuous bijection where X is compact and Y is Hausdorff, then f is a homeomorphism. Of course, $\mathbf{R}$ is not compact, so it does not apply directly. However, there is a sufficient supply of compact subsets to make the argument go through.

Recall that a continuous function

$$f\colon [a,b] \to \mathbf{R}$$

takes on an upper bound. Here is another way of putting that. If we write $\mathbf{R} = \bigcup_{n\in\mathbf{N}}(-\infty, n)$, then every function $f\colon [a,b] \to \mathbf{R}$ factors through some $(-\infty, n)$. This property is shared by other compact topological spaces.

`Proposition 9.5 (Boundedness).` *Let X, Y be topological spaces. Suppose that X is compact. Fix an increasing nested sequence $U_0 \subseteq U_1 \subseteq \cdots$ of open subsets of Y such that*

$$\bigcup_{n\in\mathbf{N}} U_n = Y.$$

Then, for every continuous function $f\colon X \to Y$, there exists an $n \in \mathbf{N}$ such that $f(X) \subseteq U_n$.

The crucial facts that make the Heine–Borel theorem go through are (a) that the topology on $\mathbf{R}$ is strongly connected to the order on $\mathbf{R}$ and (b) that the order on $\mathbf{R}$ has the least upper bound property. Both are necessary. For example, bounded closed intervals of $\mathbf{Q} \subseteq \mathbf{R}$ are not necessarily compact. For example, $[0,\sqrt{2}] \cap \mathbf{Q}$ is a closed, bounded subset which is not compact.

In fact, topology is closely bound up with order in general. We begin to discuss this below. We will return to it many times. The reader might wish to refer back to Lecture 2, especially the material on posets (Example 2.11) and preorders (Exercise 2.23).

`Construction 9.6 (Specialization preorder).` Let X be a topological space. If $x, y \in X$, write $x \lesssim y$ if every open set containing y

contains x. Equivalently, $x \lesssim y$ if $y \in \overline{\{x\}}$, i.e., y is in the closure of the singleton $\{x\}$. If $x \lesssim y$, then y is called a *specialization* of x and x is called a *generalization* of y.

`Warning 9.7.` Some authors reverse the arrow here and write $x \gtrsim y$ if y is a specialization of x.

`Lemma 9.8.` *If X is a topological space, then the specialization binary relation $\lesssim$ is a preorder on the set X.*

Proof. We have $x \lesssim x$ so it suffices to prove transitivity. If every open subset of y contains x and if every open subset of z contains y, then every open subset of z contains x. I.e., if $x \lesssim y$ and $y \lesssim z$, then $x \lesssim z$. □

`Remark 9.9.` The proof of the lemma observes that if $y \in \overline{\{x\}}$, then $\overline{\{y\}} \subseteq \overline{\{x\}}$.

`Exercise 9.10.` Let X be a topological space. Show that the specialization preorder makes X into a poset if and only if X is T_0.

`Exercise 9.11.` Let X be a topological space. Show that the specialization preorder is trivial, in the sense that $x \lesssim y$ if and only if $x = y$, if and only if X is T_1.

`Example 9.12.` If T is the Sierpiński space $\{\emptyset, \{0\}, \{0,1\}\}$, then the associated preorder is the poset $0 < 1$.

We can go the other way and start from a poset and define a topological space.

`Construction 9.13 (Poset topology).` Let $(P, \leqslant)$ be a poset. Say that a subset $U \subseteq P$ is a downset if whenever $y \in U$ and $x \leqslant y$, then $x \in U$. This is a topological space because unions and intersections of open sets are open. We will write this topological space as $P^{\leqslant}$.

`Example 9.14.` Consider the poset $0 < 1$. The associated topological space is the Sierpiński space.

`Example 9.15.` Describe the topological space associated to the poset $0 < 2$, $1 < 2$.

`Definition 9.16 (Alexandrov).` A topological space is Alexandrov if arbitrary intersections of open sets are open. Equivalently, arbitrary unions of closed sets are closed. Poset topologies are Alexandrof.

`Exercise 9.17.` Generalize Construction 9.13 to preorders. That is, given a preordered set $(P, \lesssim)$ construct a topology "the preorder topology" which agrees with the poset topology when P is a poset.

`Exercise 9.18.` If P is a poset, the poset topology on P is T_0.

`Exercise 9.19.` The real numbers are totally ordered and in particular partially ordered via $\mathbf{R}$. However, the topology $\mathbf{R}^{\leqslant}$ is not the usual topology on $\mathbf{R}$. Why not?

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