Category, space, type - Benjamin Antieau

07. Separation axioms

In a metric space, every two points are measurably distant from each other. At the other extreme, for the trivial topology X^{triv} on a set X, no two points can be distinguished topologically. There is a suite of so-called separation axioms to quantify how fine topologies are from the perspective of separating points.

Definition 7.1 (T_0). A topological space X is T_0 if for every pair of distinct points $x \neq y \in X$ there is an open subset of X which contains one but not the other. In other words, for each pair $x \neq y \in X$ there exists an open subset $U \subseteq X$ such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. Spaces which are T_0 are sometimes called *Kolmogorov spaces*; we will not use this terminology.

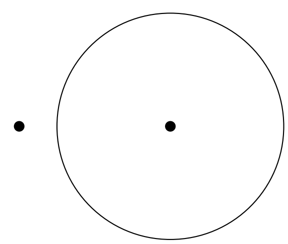

Figure 1: Diagram for T_0 spaces.

Example 7.2. Every discrete space X^{δ} is T_0.

Example 7.3. The Sierpiński space T is T_0.

Example 7.4. Every metric space T is T_0.

Example 7.5. If X is a set with at least two points, then X^{triv} is not T_0.

Remark 7.6 (Topologically indistinguishable). Let X and Y be a topological space. Say two points $x, y \in X$ are topologically indistinguishable if every open subset of X containing one contains the other. The T_0 axiom is very weak, but it is strong enough to guarantee that no two distinct points of X are topologically indistinguishable.

`Definition 7.7` (T_1). A topological space X is T_1 if for every pair of distinct points $x \neq y \in X$ there exist open subsets $x \in U \subseteq X$ and $y \in V \subseteq X$ such that $x \notin V$ and $y \notin U$.

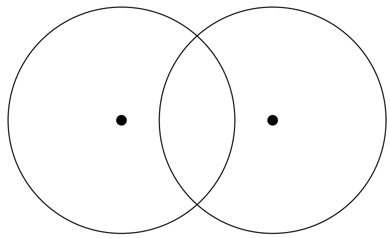

Figure 2: Diagram for T_1 spaces.

`Lemma 7.8.` *A topological space X is T_1 if and only if for every $x \in X$ the set $\{x\}$ is closed.*

Proof. Suppose that X is T_1. Let $x \in X$. For each $y \neq x$ in X, let V_y be an open subset of X which does not contain x. Let

$$V = \bigcup_{y \neq x} V_y.$$

Then, V is open and $V = X \setminus \{x\}$. Thus, $\{x\}$ is closed. Conversely, if each singleton is closed, then given $x \neq y$ in X we can set $U = X \setminus \{y\}$ and $V = X \setminus \{x\}$. Then, $x \in U$, $y \in V$, but $x \notin V$ and $y \notin U$. Thus, X is T_1. □

`Example 7.9.` Every discrete space X^δ is T_1.

`Example 7.10.` The Sierpiński space T is **not** T_1.

`Example 7.11.` Every metric space T is T_1.

`Example 7.12.` If X is a set with at least two points, then X^{triv} is not T_1.

`Exercise 7.13.` If X is T_1, then it is T_0.

`Exercise 7.14.` Suppose that X is a finite T_1 space, meaning it is a topological space which is T_1 and whose underlying set is a finite set. Show that the topology on X is discrete.

By the far the most common separation axiom which comes up in practice is the following.

Definition 7.15 (T_2 or Hausdorff). A topological space X is T_2, or Hausdorff, if for every pair of distinct points $x \neq y \in X$ there exist open subsets $x \in U \subseteq X$ and $y \in V \subseteq X$ such that $U \cap V = \emptyset$.

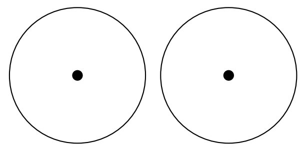

Figure 3: Diagram for T_2 spaces.

Example 7.16. Every discrete space X^δ is T_2.

Example 7.17. The Sierpiński space T is **not** T_2.

Example 7.18. Every metric space T is T_2.

Example 7.19. If X is a set with at least two points, then X^{triv} is not T_2.

Exercise 7.20. Show that if X is T_2, then it is T_1.

Example 7.21. Recall the cofinite topology on $\mathbf{N}$. The open sets are either empty or consist of all but finitely many natural numbers. This space is T_1, but it is not T_2.

Remark 7.22. There are additional separation axioms, where one starts to separate points from closed sets, from disjoint closed sets, etc. We might discuss them later.

Exercise 7.23. Show that if X and Y are Hausdorff, then so is $X \times Y$.

Exercise 7.24. Let X be a topological space. Show that the diagonal map $f\colon X \to X \times X$, defined by $f(x) = (x, x)$ for $x \in X$, is continuous. Let $\Delta = f(X)$ be the image of X, viewed as a subspace of $X \times X$ (so, a subset with the subspace topology). Show that X is Hausdorff if and only if Δ is closed in $X \times X$.

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