Category, space, type - Benjamin Antieau
06. Universal properties: the quotient topology

Dual to the subspace topology is the quotient topology.

Example 6.1 (The quotient topology). Let X be a topological space and let Y be a set. Suppose that $q\colon X \to Y$ is a surjective function. We would like to equip Y with a sensible topology. To do so, let us examine the universal property we would like Y to possess. Given a topological space Z, let $C_Y(X,Z) \subseteq C(X,Z)$ be the subset of continuous functions $f\colon X \to Z$ which factor through q as functions of sets. This means that there is a function $h\colon Y \to Z$ such that $f = h \circ q$.

Note that f factors through q if and only if for each $y \in Y$ the function f is constant on the fiber $q^{-1}(y)$.

We would like a topology on Y such that q is continuous and such that the precomposition map $C(Y,Z) \xrightarrow{h \mapsto h\circ q} C_Y(X,Z)$ is a bijection. Since we assume q is continuous, we must have that if $V \subseteq Y$ is to be open, then $q^{-1}(V)$ is open. Let $\mathcal{V} \subseteq \mathbf{P}(Y)$ be the subset of subsets $V \subseteq Y$ such that $q^{-1}(V)$ is open. We call this the quotient topology on Y and we claim it satisfies the universal property above.

By construction, $q\colon X \to Y$ is continuous with respect to the quotient topology and one can check that precomposition $C(Y,Z) \to C_Y(X,Z)$ is injective. Indeed, since $q\colon X \to Y$ is surjective, for two functions $h_0, h_1\colon Y \to Z$ which are unequal at some point y of Y, the compositions $h_0 \circ q$ and $h_1 \circ q$ are unequal at every point of the nonempty set $q^{-1}(y)$. Suppose that $f \in C_Y(X,Z)$. So, f is a continuous function $X \to Z$ which factors set theoretically through Y. Write $f = h \circ q$ for a function $Y \to Z$. Since f is continuous, if $U \subseteq Z$ is open, $f^{-1}(U)$ is open. But, this means that $f^{-1}(U) = q^{-1}(h^{-1}(U))$ is open in X. By definition of the quotient topology, this means that $h^{-1}(U)$ is open in Y. Thus, h is continuous. This implies $C(Y,Z) \cong C_Y(X,Z)$, as desired.

Example 6.2. Let X be a topological space and let $A \subseteq X$ be a subset. Let $Y = X/A$ be the result of collapsing A to a single point $a \in Y$. Thus, the quotient map $q\colon X \to Y$ is a bijection away from A and $\{a\}$ and maps A onto $\{a\}$. If $U \subseteq X$ is an open set not intersecting A, then $q(U) \subseteq Y$ is open. Suppose that $V \subseteq Y$ is an open set containing a. Then, $q^{-1}(V)$ is an open subset of

X containing A. For example, if A was open to begin with, then $\{a\} \subseteq Y$ is open.

`Example 6.3.` Consider $X = [0,1]$ and let $A = (0,1) \subseteq X$. Then, X/A has three points: $0, 1, a$. The open subsets are

$$\{\emptyset, \{a\}, \{a,0\}, \{a,1\}, \{a,0,1\}\}.$$

`Exercise 6.4.` Let $S^1 \subseteq \mathbf{R}^2$ be the unit circle. Consider the surjective map $q\colon \mathbf{R}^1 \to S^1$ obtained by $q(t) = (\cos 2\pi t, \sin 2\pi t)$. Show that the quotient topology on S^1 induced by q coincides with the subspace topology.

`Construction 6.5 (Epi-mono factorization).` If $f\colon X \to Z$ is a map of sets, then f factors uniquely as a surjection followed by an injection: $f = i \circ q$. Indeed, we can let $i\colon Y \subseteq Z$ be the image of f and let $q\colon X \to Y$ be the factorization onto its image.

If X and Z are topological spaces, we can also produce such an epi-mono factorization, but now there are in general two choices for a topology on Y. We can give it the quotient topology or the subspace topology and obtain the factorization:

$$X \xrightarrow{q} Y^{\text{quotient}} \xrightarrow{j} Y^{\text{subspace}} \xrightarrow{i} Z. \tag{1}$$

Here, j is the identity function on the set Y and it is continuous as as a map from the quotient topology to the subspace topology by continuity of f.

In practice one often wants to know that the map j appearing in (1) is a homeomorphism or in other words that the quotient and subspace topologies appearing above agree. Here are two criterian for this.

`Definition 6.6.` Let $f\colon X \to Y$ be a map of topological spaces.

(i) We say that f is open if for every open subset $U \subseteq X$ the subset $f(U)$ of Y is open.

(ii) We say that f is closed if for every closed subset $Z \subseteq X$ the subset $f(Z)$ of Y is closed.

`Example 6.7.` The function $f(x) = x^2$ from $\mathbf{R}$ to $\mathbf{R}$ is continuous, but not open: $f((-1,1)) = [0,1)$. It is closed; for example, for $0 \leqslant a \leqslant b$, $f([a,b]) = [a^2, b^2]$.

`Proposition 6.8.` *Let $f\colon X \to Y$ be a continuous surjection. If f is either open or closed, then Y has the quotient topology.*

Proof. Suppose that $U \subseteq Y$ is a subset such that $f^{-1}(U) \subseteq X$ is open. We want to know that U is open. (The converse holds by continuity of f.) However, $U = f(f^{-1}(U))$. The latter is open since f is open. The proof in the closed case is the same. □

`Remark 6.9.` If $f\colon X \to Y$ is a surjective function, the inverse images $f^{-1}(y)$ form a partition of X. Conversely, given a partition of X into disjoint subsets $\{A_y\}_{y\in Y}$, we can construct a surjective function $f\colon X \to Y$ by setting $f(x) = y$ if $x \in A_y$. Thus, "surjections are equivalent to partitions". These are also the same as equivalence relations.

`Proposition 6.10.` *Let $q\colon X \to Y$ be a surjective continuous function where Y has the quotient topology. Let $\equiv$ be the equivalence relation on X associated to q. The following conditions are equivalent:*

(a) *q is open;*

(b) *if $U \subseteq X$ is open, then the subset*

$$U^{\equiv} = \{v \in X : \text{there exists } u \in U \text{ such that } v \equiv u\}$$

is open;

(c) *if $Z \subseteq X$ is closed, then*

$$\bigcup_{y\in Y, f^{-1}(y)\subseteq Z} f^{-1}(y)$$

is closed.

Proof. If q is open, then $q(U)$ is open and hence $U^{\equiv} = q^{-1}(q(U))$ is open. Thus, (a) implies (b). Assume (b) and let $Z \subseteq X$ be closed. Let $Z_{\equiv}$ be the union appearing in (c). We want to show $Z_{\equiv}$ is closed. Let $U = X \setminus Z$ and consider $U^{\equiv}$, which is open by hypothesis. We can write $U^{\equiv} = \bigcup_{u\in U} f^{-1}(f(u))$. We claim that $Z_{\equiv}$ is disjoint from $U^{\equiv}$ and that their union is X. If x is in Z and $y \in X$ is equivalent to x, then $y \in Z_{\equiv}$ by definition and in particular, $y \in Z$. But, if $x \in U^{\equiv}$, then x is equivalent to some $z \in U$, so $z \in Z$, which is impossible. Thus, $Z_{\equiv}$ and $U^{\equiv}$ are disjoint. If $x \in X$ and x is not in $Z_\equiv$, then $f^{-1}(f(z))$ intersects U, so by transitivity of $\equiv$, $x \in U^\equiv$. Since we assume $U^\equiv$ is open, it follows that $Z_\equiv = X \setminus U^\equiv$ is closed. The proof that (c) implies (a) is left to the reader. □

`Exercise 6.11.` Prove that (c) implies (a) in Proposition 6.10.

`Remark 6.12.` Proposition 6.10 is also true with all instances of "open" replaced by "closed" and vice versa.

`Exercise 6.13.` Consider the closed unit disc $D^n \subseteq \mathbf{R}^n$ for some $n \geqslant 1$. Let $S^{n-1} \subseteq D^n$ be its boundary. Prove that the quotient space D^n/S^{n-1} is homeomorphic to S^n.

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