Category, space, type - Benjamin Antieau
05. Universal properties: first examples

We have seen two constructions of topological spaces from given ones: the subspace topology and the coproduct topology. These are examples of universal properties, which we explain in this lecture. At heart, universal properties are *categorical* properties, i.e., determined by morphisms in an ambient category.

Example 5.1 (The subspace topology). Let X be a topological space and let $A \subseteq X$ be a subset. Let Y be another topological space. Let $C_A(Y, X)$ be the set of continuous functions $f: Y \to X$ whose image is contained in the subset A. We can also view A as a subspace of X and consider $C(Y, A)$, the set of continuous functions $g: Y \to A$. Let $i: A \to X$ be the continuous function giving the embedding from A to X. Composition with i gives a function $i \circ (-): C(Y, A) \to C(Y, X)$. Evidently, the image of $i \circ (-)$ is contained in $C_A(Y, X)$, so we view it as a function $i \circ (-): C(Y, A) \to C_A(Y, X)$. Claim: this is a bijection. To see that it is injective, note that if g, g' are two continuous functions $Y \to A$ such that $i \circ g = i \circ g'$, then $g = g'$ since $i: A \to X$ is injective. To see that it is surjective, if $f: Y \to X$ has image in A, let $\overline{f}: Y \to A$ be the corresponding function on the underlying sets. Then, $\overline{f}$ is continuous: if $V \subseteq A$ is open in the subspace topology, then $V = A \cap U$ for some $U \subseteq X$ open. Since the image of f is contained in A, we have $\overline{f}^{-1}(V) = f^{-1}(U)$, which is open in Y by continuity of f.

It follows that A with its subspace topology has a universal property, which we can phrase as saying that continuous functions $Y \to A$ are "the same" as continuous functions $Y \to X$ with image in A.

We will formalize the meaning of universal properties in the next lecture. Before doing that, we give two more examples.

Example 5.2 (The coproduct topology). Let X and Y be topological spaces and consider the coproduct $X \sqcup Y$, a new topological space, as defined in Construction 4.21. The underlying set of $X \sqcup Y$ is the disjoint union of the underlying sets of X and Y. There are inclusions $i: X \to X \sqcup Y$ and $j: Y \to X \sqcup Y$. Suppose that $f: X \sqcup Y \to Z$ is a continuous function. Then, we obtain continuous functions $f \circ i: X \to Z$ and $f \circ j: Y \to Z$. Put another way, precomposition with i and j induces a function

$$(*)\colon C(X \sqcup Y, Z) \to C(X, Z) \times C(Y, Z)$$

given by

$$f \mapsto (f \circ i, f \circ j).$$

We claim that $(*)$ is a bijection. If $f \circ i = f' \circ i$, then f and f' are equal on X and if $f \circ j = f' \circ j$, then f and f' are equal on Y. Since every element of $X \sqcup Y$ is in either or X and Y, these statements together imply injectivity. Suppose that $a\colon X \to Z$ and $b\colon Y \to Z$ are continuous functions. Define $\overline{f}\colon X \sqcup Y \to Z$ be the function given by

$$\overline{f}(w) = \begin{cases} a(w) & \text{if } w \in X, \\ b(w) & \text{if } w \in Y. \end{cases}$$

The function $\overline{f}$ is continuous by definition of the subspace topology. Indeed, if $U \subseteq Z$ is open, then $\overline{f}^{-1}(U) = a^{-1}(U) \sqcup b^{-1}(U)$, the union of two open subsets of the coproduct. It follows that the function $(*)$ is surjective and hence a bijection.

Thus, $X \sqcup Y$ has a universal property: continuous functions $X \sqcup Y \to Z$ are "the same" as pairs of continuous functions $X \to Z$ and $Y \to Z$.

`Example 5.3 (The product topology).` Now, suppose that X and Y topological spaces. We want to define a topological space structure on $X \times Y$. Motivated by the previous example, we want at the very least for the projections $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ to be continuous. Since $p^{-1}(U) = U \times Y$ and $q^{-1}(V) = X \times V$, we want the subsets $U \times Y$ and $X \times V$ to be open whenever $U \subseteq X$ and $V \subseteq Y$ are open. This collection of subsets by itself does not generally make the product into a topological space because it is not closed under finite intersections or unions. Let $\mathcal{U}$ be the topology generated by the subsets of the form $U \times Y$ and $X \times V$ for $U \subseteq X$ and $V \subseteq Y$ open. We claim that the projections p and q are continuous and that the product topology has the following universal property: continuous functions $W \to X \times Y$ are "the same" as pairs of continuous functions $W \to X$ and $W \to Z$.

Continuity of the projections comes by definition of the topology. Given a continuous function $f\colon W \to X \times Y$ we can compose with p

and q to obtain continuous functions $p \circ f : W \to X$ and $q \circ f : W \to Y$. This defines a function

$$C(W, X \times Y) \to C(W, X) \times C(W, Y). \tag{1}$$

As above, we claim that $(*)$ is a bijection. We leave this as an exercise.

`Exercise 5.4.` Prove that the function in (1) is a bijection.

`Lemma 5.5.` *Let X be a set and let $\mathcal{U}' \subseteq \mathbf{P}(X)$ be a subset. There is a unique smallest topology $\mathcal{U}$ on X containing $\mathcal{U}'$.*

Proof. There is at least one topology containing $\mathcal{U}'$, namely the discrete topology $\mathbf{P}(X)$. Given any collection $\{\mathcal{U}_i\}_{i \in I}$ of topologies containing $\mathcal{U}'$ the intersection $\cap_{i \in I} \mathcal{U}_i$ is another topology containing $\mathcal{U}'$. By Zorn's lemma, it follows that there are smallest topologies containing $\mathcal{U}'$. By the intersection property above, there is in fact a unique smallest topology containing $\mathcal{U}'$. □

`Definition 5.6 (Subbasis).` Suppose that $(X, \mathcal{U})$ is a topological space and $\mathcal{U}$ is the smallest topology on X containing some $\mathcal{U}' \subseteq \mathbf{P}(X)$. Then, $\mathcal{U}'$ is said to be a subbasis for the topology on X. We also say that $\mathcal{U}$ is the topology generated by X.

`Exercise 5.7.` Let X be a topological space and let $\mathcal{U}'$ be a collection of subsets of X. Show that $\mathcal{U}'$ is a subbasis for the topology on X if and only if the following two conditions hold:

(1) the elements of $\mathcal{U}'$ are open;

(2) for every open $U \subseteq X$ and every $x \in U$, there is a finite collection $V_1, \ldots, V_n \in \mathcal{U}'$ such that $x \in V_1 \cap \cdots \cap V_n \subseteq U$.

`Definition 5.8 (Basis).` Let X be a topological space. A basis for X is a collection $\mathcal{U}'$ of open subsets satisfying the following conditions:

(a) the subsets in $\mathcal{U}'$ are open;

(b) if $U \subseteq X$ is open, then for every $x \in U$ there exists $V \in \mathcal{U}'$ such that $x \in V \subseteq U$;

(c) if $V_1, V_2 \in \mathcal{U}'$ and $x \in V_1 \cap V_2$, then there is $V_3 \in \mathcal{U}'$ such that $x \in V_3 \subseteq V_1 \cap V_2$.

`Exercise 5.9.` If X is a topological space and $\mathcal{U}'$ is a basis for the topology on X, then it is a subbasis for the topology on X.

`Exercise 5.10.` If X is a topological space and $\mathcal{U}'$ is a basis for the topology on X, then every open subset of X is a union of subsets in $\mathcal{U}'$.

`Exercise 5.11.` Suppose that X is a topological space and $\mathcal{U}'$ is a subbasis for the topology on X. Let $\mathcal{U}'' \subseteq \mathbf{P}(X)$ be the collection of subsets obtained by intersecting finitely many subsets in $\mathcal{U}'$. Show that $\mathcal{U}''$ is a basis for the topology on X.

`Example 5.12.` A subbasis for the usual topology on $\mathbf{R}$ is given by the collection of rays $(-\infty, a)$ and (b, ∞) for $a, b \in \mathbf{R}$. A basis for the usual topology on $\mathbf{R}$ is given by the collection of open intervals (a, b).

`Remark 5.13.` It follows that if $\mathcal{U}'$ is a subbasis for a topology $\mathcal{U}$ on X, then every open subset U of X is of the form

$$U = \bigcup_{i \in I} \left(\bigcap_{j=1}^{a_i} U_{ij} \right)$$

for some indexing set I, some integers $a_i \geqslant 1$, and some $U_{ij} \in \mathcal{U}'$. In other words, every open is a union of finite intersections of subbasis elements.

`Remark 5.14 (Return to the product topology).` Let X and Y be topological spaces. The collection $\mathcal{U}'$ of open sets $U \times Y$ and $X \times V$, $U \subseteq X$ and $V \subseteq Y$ open, forms a subbasis for the product topology on $X \times Y$. Equivalently, it generates the product topology. A basis for the product topology is given by $\mathcal{U}''$, the collection of subsets of the form $U \times V$ for $U \subseteq X$ and $V \subseteq Y$ open.

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