Category, space, type - Benjamin Antieau
04. Subspaces and connectedness

Suggested reading. Read Section 23 from Munkres; review exercises 14.1 and 14.2.

Sets can be broken up into disjoint subsets at will. They do not have internal structure to preserve.

Lemma 4.1. *Let $(X, \mathcal{U})$ be a topological space and let $Y \subseteq X$ be a subset. Consider the collection $\mathcal{V}$ of subsets of Y of the form $U \cap Y$ where $U \subseteq X$ is open. Then, $(Y, \mathcal{V})$ is a topological space and the inclusion $i\colon Y \hookrightarrow X$ is continuous.*

Proof. We have $i^{-1}(U) = U \cap Y$, so the second statement follows immediately once we have established the first. We have $\emptyset \cap Y = \emptyset$ and $X \cap Y = Y$, so that $\emptyset, Y \in \mathcal{V}$. Given a collection $\{V_i\}_{i \in I} \subseteq \mathcal{V}$, for each $i \in I$, choose U_i open in X such that $V_i = U_i \cap Y$. Then, $\cup_{i \in I} V_i = \cup_{i \in I}(U_i \cap Y) = (\cup_{i \in I} U_i) \cap Y$, so that $\cup_{i \in I} V_i \in \mathcal{V}$. We also have $\cap_{i \in I} V_i = \cap_{i \in I}(U_i \cap Y) = (\cap_{i \in I} U_i) \cap Y$. If I is finite, then $\cap_{i \in I} U_i$ is open, so $\cap_{i \in I} V_i \in \mathcal{V}$. This completes the proof. □

Definition 4.2 (Subspace). If X is a topological space a subspace is a subset $Y \subseteq X$ equipped with the subspace topology defined in Lemma 4.1.

Warning 4.3. Suppose that Y is a subspace of a topological space X. Let $A \subseteq Y$ be a subset. Whether A is open or not depends, in general, on whether we view it as a subset of Y or X. For example, Y is always open in itself, but might or might not be open in X.

Exercise 4.4. Let X be a topological space and let $Y \subseteq X$ be a subspace. Show that the following conditions are equivalent:

(i) a subset $A \subseteq Y$ is open in Y if and only if it is open in X;

(ii) Y is open in X.

Exercise 4.5. Let X be a topological space and let $Y \subseteq X$ be a subspace. Show that $A \subseteq Y$ is closed in Y if and only there is a closed subset $Z \subseteq X$ such that $Z \cap Y = A$.

Example 4.6. A subspace of a discrete topological space is discrete.

`Example 4.7.` A subspace of a topological space with the trivial topology has the trivial topology.

`Exercise 4.8.` Suppose that $\mathbf{N}$ is the set of natural numbers with the cofinite topology of Example 1.14. Let $F \subseteq \mathbf{N}$ is a finite subspace. Show that the topology on F is discrete.

Now, we come to our first major topological property, connectedness.

`Definition 4.9.` Let X be a topological space. A subset $U \subseteq X$ is clopen if it is open and closed.

`Remark 4.10.` If X is a topological space, then $\emptyset$ and X are clopen subsets of X.

`Example 4.11.` Let X^δ be a discrete topological space. Then, if $x \in X$, the subset $\{x\} \subseteq X$ is clopen.

`Definition 4.12 (Connectedness).` Say that a topological space X is connected if it has no nonempty proper clopen subsets.

Equivalently, X is connected if its only clopen subsets are $\emptyset, X$.

`Remark 4.13.` Connectedness is a topological property of topological spaces. If X and Y are homeomorphic topological spaces, then X is connected if and only if Y is connected.

`Example 4.14.` The empty set is connected. This disagrees with the convention of *The Stacks Project.*

`Example 4.15.` If X is a set, then X^{triv} is connected.

`Example 4.16.` The Sierpiński space T is connected.

`Example 4.17.` If X is a set with at least two elements, then X^δ is not connected.

`Lemma 4.18.` *The topological space* $\mathbf{R}$ *is connected. Similarly, every interval* (a, b) *or* $[a, b)$ *or* $(a, b]$ *or* $[a, b]$ *for* $-\infty \leqslant a \leqslant b \leqslant \infty$ *is connected.*

Proof. We give the proof for $\mathbf{R}$. Suppose that $U \subseteq \mathbf{R}$ is a nonempty proper clopen subset. Let $V = \mathbf{R} \setminus U$. Then, V is also proper, nonempty, and clopen. Consider the function $f\colon \mathbf{R} \to \{0, 1\}^\delta$ such that $f(u) = 0$ for $u \in U$ and $f(v) = 1$ for $v \in V$. Then, f is continuous. The inclusion $i\colon \{0,1\}^\delta \subseteq \mathbf{R}$ is continuous. It follows that $i \circ f\colon \mathbf{R} \to \mathbf{R}$ is continuous. The intermediate value theorem (which depends on the fact that every nonempty bounded above subset of $\mathbf{R}$ has a least upper bound) leads to a contradiction. □

`Exercise 4.19.` Let $\mathbf{Q} \subseteq \mathbf{R}$ be the subspace of rational numbers. Determine whether or not $\mathbf{Q}$ is connected.

`Exercise 4.20.` The cofinite topology on $\mathbf{N}$ is connected.

`Construction 4.21 (The coproduct topology).` Let X and Y be topological spaces. Let $Z = X \sqcup Y$ be the disjoint union of X and Y. We define a topology on Z by saying that $U \subseteq Z$ is open if and only if $U \cap X$ and $U \cap Y$ are open. In other words, the open subsets of Z are precisely those which can be written as $V \sqcup W$ where $V \subseteq X$ and $W \subseteq X$ are open. Note that the inclusions $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$ are continuous.

`Lemma 4.22.` *Suppose that X is* not *connected. Then, there are disjoint nonempty clopen subspaces $U, V \subseteq X$ such that the natural map $U \sqcup V \to X$ is a homeomorphism.*

Proof. As X is not connected, there is a nonempty proper clopen subset $U \subseteq X$ which we view as a subspace. Let $V = X \setminus U$. Since U is a proper subset, V is nonempty. It is also clopen. As sets, we have $X = U \sqcup V$. The natural inclusion $i\colon U \sqcup V \to X$ is continuous by definition of the coproduct topology. It is also a bijection. Let $j\colon X \to U \sqcup V$ be the inverse function. Let $W \subseteq U \sqcup V$ be an open subset. By definition, this means that $W \cap U \subseteq U$ is open and $W \cap V \subseteq V$ is open. Now, $j^{-1}(W) = W = (W \cap U) \cup (W \cap V)$. Since $W \cap U$ is open in U and U is open in X, we have that $W \cap U$ is open in X. Similarly, $W \cap V$ is open in X. Thus, the union $j^{-1}(W)$ is open in X. Thus, j is continuous and hence i is a homeomorphism. □

`Example 4.23.` [Invariance of domain I] The topological space $\mathbf{R} \setminus \{0\}$ is not connected. Indeed, $(0,\infty)$ and $(-\infty, 0)$ are nonempty clopen disjoint subspaces whose union is $\mathbf{R} \setminus \{0\}$. On the other hand, $\mathbf{R}^2 \setminus \{0\}$ is connected. Suppose note and that $\mathbf{R}^2 \setminus \{0\} = U \sqcup V$ is a clopen decomposition with $x \in U$ and $y \in V$. We can find $\gamma\colon [0,1] \to \mathbf{R}^2 \setminus \{0\}$ a continuous function such that $\gamma(0) = x$ and $\gamma(1) = y$. Then, $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are disjoint clopen subsets of $[0,1]$ containing x and y, respectively. In particular, they are nonempty. Thus, $[0,1]$

is disconnected, which contradicts Lemma 4.18. It follows that $\mathbf{R}$ and $\mathbf{R}^2$ are not homeomorphic. Indeed, if they were, then possibly after applying a suitable translation we could assume the existence of a homeomorphism $f\colon \mathbf{R} \to \mathbf{R}^2$ such that $f(0) = 0$. Then, f would restrict to a homeomorphism $f'\colon (\mathbf{R} \setminus \{0\}) \to (\mathbf{R}^2 \setminus \{0\})$, a contradiction.

`Definition 4.24 (Path-connectedness).` Let X be a topological space. A path in X is a continuous function $\gamma\colon [0,1] \to X$. If $x, y \in X$, a path in X from x to y is a path γ such that $\gamma(0) = x$ and $\gamma(1) = y$. We say that X is path-connected if for every $x, y \in X$ there is a path from x to y.

`Example 4.25.` Show that the Sierpiński space T is path-connected.

`Exercise 4.26.` Let X be a topological space. For $x, y \in X$, write $x \sim y$ if there is path in X from x to y. Prove that $\sim$ is an equivalence relation on the set of points of X.

The argument in Example 4.23 implies the following lemma.

`Lemma 4.27.` *A path-connected topological space is connected.*

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