Category, space, type - Benjamin Antieau
02. Categories

We introduce categories in order to study the relationships between different topological spaces. The word category should evoke the idea of isolating a certain type of object, in our case mathematical objects. The most basic type of object is that of a set. Besides sets themselves, we also have functions between sets and compositions of such functions and these satisfy certain properties. A category is an abstraction of this relationship between sets and the functions between them.

Definition 2.1 (Categories). A category $\mathcal{C}$ consists of the following data:

(i) a class of objects $\mathrm{Ob}(\mathcal{C})$;

(ii) for every pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a set $\mathrm{Hom}_{\mathcal{C}}(x, y)$, whose elements are the morphisms from x to y;

(iii) for every triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a composition map

$$\mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \xrightarrow{(g,f)\mapsto g\circ f} \mathrm{Hom}_{\mathcal{C}}(x, z);$$

(iv) for every $x \in \mathrm{Ob}(\mathcal{C})$ a distinguished morphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$.

This data is required to satisfy the following conditions:

(a) for every pair $x, y \in \mathrm{Ob}(\mathcal{C})$ and every $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ one has $f = f \circ \mathrm{id}_x = \mathrm{id}_y \circ f$;

(b) for every quadruple w, x, y, z and every triple of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(w, x)$, $g \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, $h \in \mathrm{Hom}_{\mathcal{C}}(y, z)$, the compositions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are equal in $\mathrm{Hom}_{\mathcal{C}}(w, z)$.

Condition (a) is called unitality while (b) is associativity.

Notation 2.2. If $\mathcal{C}$ is a category, we will simply write $x \in \mathcal{C}$ if we mean that $x \in \mathrm{Ob}(\mathcal{C})$. We write $f\colon x \to y$ or $x \xrightarrow{f} y$ to denote that $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$.

Example 2.3. Let **Set** denote the category of sets. The class of objects is the proper class of all sets (see Remark 2.4); the morphisms are functions between sets; composition is composition of functions. Given a set X, the distinguished function $\mathrm{id}_X \in \mathrm{Hom}_{\mathbf{Set}}(X, X)$ is the usual identity function defined by $f(x) = x$ for all $x \in X$.

`Remark 2.4.` We are not going to get bogged down in the foundations of set theory. Russell's paradox is that if one allows for a set of all sets S one can obtain a contradiction by considering the set $C \subseteq S$ of sets which contain themselves. Ponder the question of whether $C \in C$. In order to make sense of a notion of a collection of all sets without falling prey to Russell's paradox, set theorists have introduced a notion of classes, which include sets and classes which are not sets; the latter are called proper classes. One axiomatization is due to von Neumann–Bernays–Gödel (NBG); it is an extension of the standard "Zermelo–Fraenkel with the axiom of choice" approach to set theory (ZFC). The basic relation is still containment $\in$. The main thing to remember is that if $x \in y$, then x must be a set; y can be a set or a proper class. There is a proper class of all sets.

What we have defined above is what is sometimes called a *locally small* category. There is a variant where the $\mathrm{Hom}_{\mathcal{C}}(x, y)$ are allowed to be proper classes. We will not use this generality in these notes, so for us a category will always mean a locally small category, as given in Definition 2.1. On the other hand, if $\mathrm{Ob}(\mathcal{C})$ is a set as opposed to a proper class, then we say that $\mathcal{C}$ is a small category. If $\mathcal{C}$ is not small, we say it is large. The category **Set** is large.

`Example 2.5.` The empty set $\emptyset$ forms a category, also denoted $\emptyset$, in a natural way.

`Example 2.6.` Let **Group** denote the large category of groups. The class of objects is the proper class of all groups, the morphisms are group homomorphisms, and composition is as usual.

`Example 2.7.` Let k be a field. Let $\mathbf{Vect}_k$ be the large category of k-vector spaces. The class of objects is the proper class of all k-vector spaces and the morphisms are k-linear transformations.

`Example 2.8.` Let $\mathcal{C}$ be the small category whose objects are natural numbers $n \geqslant 0$ and where $\mathrm{Hom}_{\mathcal{C}}(m, n)$ is equal to the set of infinitely differentiable functions $\mathbf{R}^m \to \mathbf{R}^n$. Composition is given by composition of functions.

There are more exotic examples.

`Exercise 2.9.` Let M be a monoid. Recall that this is a set equipped with an associative, unital binary operation $M \times M \to M$. Define a small category $\mathrm{B}M$ with a single object $*$ where

$\mathrm{Hom}_{\mathrm{B}M}(*,*) = M$ and where composition is given by the monoid operation.

`Exercise 2.10.` Show conversely that if $\mathcal{C}$ is a category and $x \in \mathcal{C}$, then composition makes $\mathrm{Hom}_{\mathcal{C}}(x,x)$ into a monoid with unit id_x.

`Example 2.11 (Posets).` Let $(P, \leqslant)$ be a poset, i.e., a set with a binary relation which is reflexive, transitive, and antisymmetric. We can view P as a category as follows. Given $x, y \in P$, let

$$\mathrm{Hom}_P(x,y) = \begin{cases} * & \text{if } x \leqslant y, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus, given $x, y \in P$, there is at most one map from x to y, defined if and only if $x \leqslant y$. Composition is well-defined thanks to the transitivity of $\leqslant$ for posets.

`Definition 2.12 (Isomorphism).` There is a notion of 'sameness' internal to any category $\mathcal{C}$. A map $f\colon x \to y$ in $\mathcal{C}$ is an isomorphism if there is a map $g\colon y \to x$ such that $g \circ f = \mathrm{id}_x$ and $f \circ g = \mathrm{id}_y$. If an isomorphism $f\colon x \to y$ exists, then x and y are said to be isomorphic; this is written symbolically as $x \cong y$.

Isomorphic objects are to be viewed as having the same properties in the same way that the sets $\{1,2,3\}$ and $\{a,b,c\}$, while not equal, behave identically in the category **Set**.

`Exercise 2.13.` A map $f\colon X \to Y$ of sets is an isomorphism (i.e., in the category **Set**) if and only if it is a bijection.

`Exercise 2.14.` Let $\mathcal{C}$ be a category.

(i) Show that every object $x \in \mathcal{C}$ is isomorphic to itself.

(ii) Show that if $x \cong y$ and $y \cong z$, then $x \cong z$.

(iii) Show that if $f\colon x \to y$ is an isomorphism, then if $g, h\colon y \to x$ satisfy $g \circ f = h \circ f = \mathrm{id}_x$ and $f \circ g = f \circ h = \mathrm{id}_y$, then $g = h$. Thus, inverses, when they exist, are unique. We write f^{-1} for the inverse, when it exists.

`Exercise 2.15.` Show that if $x \cong y$ in a poset P, then $x = y$.

`Definition 2.16 (Groupoids).` Let $\mathcal{C}$ be a category. If every morphism in $\mathcal{C}$ is an isomorphism, then $\mathcal{C}$ is called a groupoid.

The terminology is motivated by the following example.

`Exercise 2.17.` Let M be a monoid. Show that $\mathrm{B}M$ is a groupoid if and only if M is a group.

`Exercise 2.18.` Characterize the posets P which are groupoids.

`Example 2.19 (Walking isomorphism).` Consider the small category $\mathcal{C}$ with set of objects $\{0, 1\}$ and where all Hom sets consist of a single element. This is a groupoid whose nonidentity morphisms can be displayed as follows:

$$0 \rightleftarrows 1.$$

`Definition 2.20 (Thin categories).` A category $\mathcal{C}$ is thin if

$$\mathrm{Hom}_{\mathcal{C}}(x, y)$$

consists of at most 1 element for all $x, y \in \mathcal{C}$. A poset when viewed as a category is thin. So is the walking isomorphism. The category **Set** is not thin.

`Example 2.21.` Let M be a monoid. The category $\mathrm{B}M$ is thin if and only if $M = *$ is the trivial monoid.

We see that categories can be used both as an organizing principle, as in the cases of **Set**, **Group**, or $\mathbf{Vect}_k$, but they can also be used as combinatorial objects as in the case of posets.

`Lemma 2.22.` *Let $\mathcal{C}$ be a thin small category. For $x, y \in \mathcal{C}$, say that $x \lesssim y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is nonempty. This relation makes $\mathrm{Ob}(\mathcal{C})$ into a poset if and only if $\mathcal{C}$ has no non-identity isomorphisms.*

Proof. Suppose that the relation $x \lesssim y$ makes $\mathrm{Ob}(\mathcal{C})$ into a poset. Then, if $x \lesssim y$ and $y \lesssim x$, then $x = y$. If $f\colon x \to y$ is a non-identity isomorphism with inverse g, then $x \neq y$ but $x \lesssim y$ and $y \lesssim x$, a contradiction.

Conversely, suppose that $\mathcal{C}$ has no non-identity isomorphisms. The relation $\lesssim$ is reflexive since $\mathrm{Hom}_{\mathcal{C}}(x, x)$ contains id_x for all $x \in \mathcal{C}$. It is transitive since if $\mathrm{Hom}_{\mathcal{C}}(y, z)$ is nonempty and $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is nonempty, so is $\mathrm{Hom}_{\mathcal{C}}(x, z)$ by composition. It is antisymmetric because there are no non-identity isomorphisms. $\square$

`Exercise 2.23.` A preorder is a set P equipped with a binary relation $\lesssim$ which is reflexive and transitive. Show that every preorder gives rise to a small thin category and conversely every small thin category induces the structure of a preorder on its set of objects.

`Example 2.24.` The walking isomorphism is the basic example of a preorder which is not a poset.

Department of Mathematics, Northwestern University
antieau@northwestern.edu